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Reversibility of Some Chordal SLE($\kappa; \rho$) Traces

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Abstract We prove that, for $\kappa \in (0,4)$ and $\rho \geq (\kappa-4)/2$, the chordal SLE($\kappa;\rho$) trace started from $(0;0^+)$ or $(0;0^-)$ satisfies the reversibility property. And we obtain the equation for the reversal of the chordal SLE($\kappa;\rho$) trace started from $(0;b_0)$, where $b_0>0$.

Keywords SLE · Reversibility · Coupling technique

1 Introduction

In the proof of the reversibility of the SLE(κ) trace [9], where $\kappa \in (0,4]$, a new technique was developed to construct a coupling of two SLE(κ) traces, such that in that coupling, the images of the two traces coincide, and the directions of the two traces are opposite. That technique was then used to prove the Duplantier's duality conjecture [10, 11]. Comparing Theorem 5.4 in [10] with Julien Dubédat's Conjecture 2 in [1], the author proposed the following conjecture in [10].

Conjecture 1 *Let* $\beta_0(t)$, $0 \leq t < \infty$, *be a chordal* SLE($\kappa;\rho_+,\rho_-$) *trace started from* $(0;0^+,0^-)$, *where* $\kappa \in (0,4)$ *and* $\rho_+,\rho_- \geq (\kappa-4)/2$. *Let* $W_0(z) = 1/\overline{z}$. *Then after a time-change*, $(W_0(\beta_0(1/t)))$, $0<t<\infty$, *has the same distribution as* $(\beta_0(t))$, $0<t<\infty$.

It's already known that this conjecture holds in some special cases. If $\rho_+ = \rho_- = 0$, then β_0 is a standard SLE(κ) trace, and the result follows from [9]. If $\kappa = 0$, then β_0 is a half line from 0 to ∞, which is a trivial case. If $\kappa = 4$, then it follows from the convergence of the discrete Gaussian free field contour line [5]; and it is also a special case of Theorem 5.5 in [10]. The motivation of the current paper is to prove the above conjecture. We will only prove part of it, that is, the case when ρ_+ or ρ_- equals to 0. If, for example, $\rho_- = 0$, then β_0 reduces to a chordal SLE($\kappa;\rho_+$) trace started from $(0;0^+)$. The main theorem of this paper is the following.

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Theorem 1.1 *Let $\kappa \in (0,4)$ and $\rho \ge (\kappa-4)/2$. Suppose $\beta_0(t)$, $0 \le t < \infty$, is a chordal* $\mathrm{SLE}(\kappa;\rho)$ *trace started from* $(0;0^\sigma)$, *where $\sigma \in \{+,-\}$. Let $W_0(z) = 1/\overline{z}$. Then after a time-change, $W_0(\beta_0(1/t))$, $0 < t < \infty$, has the same distribution as $\beta_0(t)$, $0 < t < \infty$.*

We will see that Theorem 1.1 here and Theorem 5.4 in [10] imply Dubédat's conjecture. Besides the special cases that $\rho = 0$, $\kappa = 0$ or 4, the above theorem is also known to be true in the case that $\kappa = 8/3$. This follows from [3] because the image of β_0 satisfies the left-sided or right-sided restriction property with exponent depending on ρ, and the one-sided restriction measure is invariant under the map $W_0(z) = 1/\overline{z}$.

The proof of Theorem 1.1 will be completed in the last section. We will use the technique used in [9] and [10]. The new difficulty here is that when applying the above technique, we need some information about the "middle" part of the curve β_0. This means that given a stopping time $T_1 > 0$ and a "backward" stopping time $T_2 < \infty$ with $T_1 < T_2$, we need to know the conditional distribution of $\beta_0(t)$, $T_1 < t < T_2$, given the curves $\beta_0((0;T_1])$ and $\beta_0([T_2;\infty))$. This is known in some special cases. If β_0 is a standard chordal $\mathrm{SLE}(\kappa)$ trace, which corresponds to the case that $\rho = 0$, then $\beta_0(t)$, $T_1 < t < T_2$, is a time-change of a chordal $\mathrm{SLE}(\kappa)$ trace in $\mathbb{H} \setminus (\beta_0((0;T_1]) \cup \beta_0([T_2;\infty)))$ from $\beta_0(T_1)$ to $\beta_0(T_2)$. If $\kappa = 4$, from the proof of Theorem 5.5 in [10], we see that $\beta_0(t)$, $T_1 < t < T_2$, is a time-change of a generic $\mathrm{SLE}(\kappa;\rho)$ trace in $\mathbb{H} \setminus (\beta_0((0;T_1]) \cup \beta_0([T_2;\infty)))$. In the general case, as we will see, the conditional distribution of $\beta_0(t)$, $T_1 < t < T_2$, is complicated. To describe this middle part of β_0, we will use hypergeometric functions to define a new kind of SLE-type processes, which are called intermediate $\mathrm{SLE}(\kappa;\rho)$ processes. These new SLE-type processes will also be used to describe the reversal of an $\mathrm{SLE}(\kappa;\rho)$ trace whose force point is not degenerate. This is Theorem 1.2 below, whose proof will also be completed in the last section.

Theorem 1.2 *Suppose $\beta_0(t)$, $0 \le t < \infty$, is a chordal* $\mathrm{SLE}(\kappa;\rho)$ *trace started from* $(0;b_0)$ *with $b_0 > 0$. Let $W_0(z) = 1/\overline{z}$. Then after a time-change, $W_0(\beta_0(1/t))$, $0 < t < \infty$, has the same distribution as a degenerate intermediate* $\mathrm{SLE}(\kappa;\rho)$ *trace with force points 0^+ and $1/b_0$.*

The current paper will frequently use results from [9] and [10]. The reader is suggested to have copies of those two papers by hand for convenience.

After finishing the first version of this paper, the author noticed that Corollary 9 in [2] is equivalent to Theorem 1.1 here. It seems to the author that some important details are omitted in [2]. The proofs in this paper will be completed, and contain all details. And the approach of this paper is somewhat different from that in [2].

2 Preliminary

If H is a bounded and relatively closed subset of $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and $\mathbb{H} \setminus H$ is simply connected, then we call H a hull in $\mathbb{H}$ w.r.t. ∞. For such H, there is φ_H that maps $\mathbb{H} \setminus H$ conformally onto $\mathbb{H}$, and satisfies $\varphi_H(z) = z + \frac{c}{z} + O(\frac{1}{z^2})$ as $z \to \infty$, where $c = \mathrm{hcap}(H) \ge 0$ is called the half-plane capacity of H. A hull H with $\mathrm{hcap}(H) = c$ has diameter at least $\sqrt{c}$. If $H_1 \subset H_2$ are hulls in $\mathbb{H}$ w.r.t. ∞, then $H_2/H_1 := \varphi_{H_1}(H_2 \setminus H_1)$ is also a hull in $\mathbb{H}$ w.r.t. ∞, and we have $\varphi_{H_2} = \varphi_{H_2/H_2} \circ \varphi_{H_1}$.

For a real interval I, we use $C(I)$ to denote the space of real continuous functions on I. For $T>0$ and $\xi\in C([0,T))$, the chordal Loewner equation driven by ξ is

$$\partial_t\varphi(t,z)=\frac{2}{\varphi(t,z)-\xi(t)},\quad \varphi(0,z)=z. \tag{2.1}$$

For $0\le t<T$, let $K(t)$ be the set of $z\in\mathbb{H}$ such that the solution $\varphi(s,z)$ blows up before or at time t. Then each $K(t)$ is a hull in $\mathbb{H}$ w.r.t. ∞, $\operatorname{hcap}(K(t))=2t$, and $\varphi(t,\cdot)=\varphi_{K(t)}$. We call $K(t)$ and $\varphi(t,\cdot)$, $0\le t<T$, the chordal Loewner hulls and maps, respectively, driven by ξ.

Let $B(t)$, $0\le t<\infty$, be a (standard) Brownian motion. Let $\kappa>0$. Then $K(t)$ and $\varphi(t,\cdot)$, $0\le t<\infty$, driven by $\xi(t)=\sqrt{\kappa}B(t)$, $0\le t<\infty$, are called the standard chordal SLE(κ) hulls and maps, respectively. It is known [4, 8] that almost surely for any $t\in[0,\infty)$,

$$\beta(t):=\lim_{\mathbb{H}\ni z\to\xi(t)}\varphi(t,\cdot)^{-1}(z) \tag{2.2}$$

exists, and $\beta(t)$, $0\le t<\infty$, is a continuous curve in $\overline{\mathbb{H}}$. Moreover, if $\kappa\in(0,4]$ then β is a simple curve, which intersects $\mathbb{R}$ only at the initial point, and for any $t\ge0$, $K(t)=\beta((0,t])$; if $\kappa>4$ then β is not simple, and intersects $\mathbb{R}$ at infinitely many points; and in general, $\mathbb{H}\setminus K(t)$ is the unbounded component of $\mathbb{H}\setminus\beta((0,t])$ for any $t\ge0$. Such β is called a standard chordal SLE(κ) trace.

If $(\xi(t))$ is a semi-martingale with $d\langle\xi\rangle_t=\kappa\,dt$ for some $\kappa>0$, then from the Girsanov's theorem ([7]) and the existence of standard chordal SLE(κ) trace, we see that almost surely for any $t\in[0,T)$, $\beta(t)$ defined by (2.2) exists, and has the same property as a standard chordal SLE(κ) trace (depending on the value of κ) as described in the last paragraph.

Let $\kappa>0$, $N\in\mathbb{N}$, $\vec{\rho}=(\rho_1,\dots,\rho_N)\in\mathbb{R}^N$, $x_0\in\mathbb{R}$, and $\vec{p}=(p_1,\dots,p_N)\in(\widehat{\mathbb{R}}\setminus\{x_0\})^N$, where $\widehat{\mathbb{R}}=\mathbb{R}\cup\{\infty\}$ is a circle. Let $B(t)$ be a Brownian motion, which generates a filtration $(\mathcal{F}_t)$. Let $\xi(t)$ and $p_m(t)$, $1\le m\le N$, $0\le t<T$, be the maximal solutions to the SDE:

$$\begin{cases} d\xi(t)=\sqrt{\kappa}\,dB(t)+\sum_{m=1}^N\frac{\rho_m\,dt}{\xi(t)-p_m(t)}\\ dp_m(t)=\frac{2dt}{p_m(t)-\xi(t)},\quad 1\le m\le N,\end{cases} \tag{2.3}$$

with initial values

$$\xi(0)=x_0,\qquad p_m(0)=p_m,\quad 1\le m\le N.$$

The meaning of the maximal solutions is that $[0,T)$ is the maximal interval of the solution. Here if some $p_m=\infty$ then $p_m(t)=\infty$ and $\frac{\rho_m}{\xi(t)-p_m(t)}=0$ for all $t\ge0$, so p_m has no effect on the equation. Let $K(t)$, $0\le t<T$, be the chordal Loewner hulls driven by ξ. Then we call $K(t)$, $0\le t<T$, a chordal SLE($\kappa;\rho_1,\dots,\rho_N$) or SLE($\kappa;\vec{\rho}$) process started from $(x_0;p_1,\dots,p_N)$ or $(x_0;\vec{p})$. It is known that $(\xi(t))$ is an $(\mathcal{F}_t)$-semi-martingale with $d\langle\xi\rangle_t=\kappa dt$. So the chordal Loewner trace $\beta(t)$, $0\le t<T$, driven by ξ exists, and is called a chordal SLE($\kappa;\vec{\rho}$) trace started from $(x_0;\vec{p})$. These p_m's and ρ_m's are called the force points and forces, respectively.

The chordal SLE($\kappa;\vec{\rho}$) processes defined above are of generic cases. We now introduce degenerate SLE($\kappa;\vec{\rho}$) processes, where one of the force points takes value x_0^+ or x_0^-, or two of the force points take values x_0^+ and x_0^-, respectively. The force point x_0^+ or x_0^- is called a degenerate force point. The definitions are as follows. Suppose $p_1=x_0^+$ is the only degenerate force point. Let $\xi(t)$ and $p_m(t)$, $1\le k\le N$, $0<t<T$, be the maximal solution

to (2.3) with initial values

$$\xi(0) = p_1(0) = x_0, \qquad p_k(0) = p_k, \quad 2 \le k \le N.$$

Moreover, we require that

$$p_1(t) > \xi(t), \quad 0 < t < T. \tag{2.4}$$

It is known that the solution exists, and $(\xi(t))$ is also an $(\mathcal{F}_t)$-semi-martingale with $d\langle \xi \rangle_t = \kappa\, dt$. The chordal Loewner trace driven by $\xi(t)$, $0 \le t < T$, is called a chordal $\mathrm{SLE}(\kappa; \rho_1, \dots, \rho_N)$ trace started from $(x_0; x_0^+, p_2, \dots, p_N)$. If the ">" in (2.4) is replaced by "<", then we get a chordal $\mathrm{SLE}(\kappa; \rho_1, \dots, \rho_N)$ trace started from $(x_0; x_0^-, p_2, \dots, p_N)$. If the only degenerate force points are $p_1 = x_0^+$ and $p_2 = x_0^-$, let $\xi(t)$ and $p_k(t)$, $1 \le k \le N$, $0 < t < T$, be the maximal solution to (2.3) with initial values

$$\xi(0) = p_1(0) = p_2(0) = x_0, \qquad p_k(0) = p_k, \quad 3 \le k \le N$$

such that

$$p_1(t) > \xi(t) > p_2(t), \quad 0 < t < T.$$

The chordal Loewner trace driven by $\xi(t)$, $0 \le t < T$, is called a chordal $\mathrm{SLE}(\kappa; \rho_1, \dots, \rho_N)$ trace started from $(x_0; x_0^+, x_0^-, p_3, \dots, p_N)$.

For $1 \le m \le N$, the function $p_m(t)$, $0 \le t < T$, is called the force point function started from p_m. Each force point function is determined by its initial point p_m and the driving function $\xi(t)$ as follows. Let $\varphi(t, \cdot)$, $0 \le t < T$, be the chordal Loewner maps driven by ξ. If p_m is not degenerate, then from (2.1), we have $p_m(t) = \varphi(t, p_m)$, $0 \le t < T$. If $p_m = x_0^\sigma$, $\sigma \in \{+, -\}$, is degenerate, then it is not difficult to see that $p_m(t) = \lim_{x \to x_0^\sigma} \varphi(t, x)$.

The following lemma is a special case of Lemma 2.1 in [10].

Lemma 2.1 *Suppose* $\kappa \in (0, 4]$ *and* $\vec{\rho} = (\rho_1, \dots, \rho_N)$ *with* $\sum_{m=1}^N \rho_m = \kappa - 6$. *For* $j = 1, 2$, *let* $K_j(t)$, $0 \le t < T_j$, *be a generic or degenerate chordal* $\mathrm{SLE}(\kappa; \vec{\rho})$ *process started from* $(x_j; \vec{p}_j)$, *where* $\vec{p}_j = (p_{j,1}, \dots, p_{j,N})$, $j = 1, 2$. *Suppose* W *is a conformal or conjugate conformal map from* $\mathbb{H}$ *onto* $\mathbb{H}$ *such that* $W(x_1) = x_2$ *and* $W(p_{1,m}) = p_{2,m}$, $1 \le m \le N$. *Then* $(W(K_1(t)), 0 \le t < T_1)$ *has the same law as* $(K_2(t), 0 \le t < T_2)$ *up to a time-change. A similar result holds for the traces.*

The following lemma is a special case of Theorem 3.2 in [10].

Lemma 2.2 *Suppose* $\kappa \in (0, 4]$, $\rho \ge (\kappa - 4)/2$, *and* $\beta(t)$, $0 \le t < \infty$, *is a chordal* $\mathrm{SLE}(\kappa; \rho)$ *trace started from* $(0; 0^\sigma)$, *where* $\sigma \in \{+, -\}$. *Then a.s.* $\lim_{t \to \infty} \beta(t) = \infty$.

From Lemma 2.1 and Lemma 2.2, we obtain the following lemma.

Lemma 2.3 *Let* $\kappa \in (0, 4]$, $\rho \ge (\kappa - 4)/2$, *and* $x_1 \ne x_2 \in \mathbb{R}$. *Suppose* $\beta(t)$, $0 \le t < T$, *is a chordal* $\mathrm{SLE}(\kappa; \rho, \kappa - 6 - \rho)$ *trace started from* $(x_1; x_1^\sigma, x_2)$, *where* $\sigma \in \{+, -\}$. *Then a.s.* $\lim_{t \to T^-} \beta(t) = x_2$.

Proof Let $\beta_0(t)$, $0 \le t < \infty$, be a chordal $\mathrm{SLE}(\kappa; \rho)$ trace started from $(0; 0^+)$. From Lemma 2.2, a.s. $\lim_{t \to \infty} \beta_0(t) = \infty$. We may find W that maps $\mathbb{H}$ conformally or conjugate conformally onto $\mathbb{H}$ such that $W(0) = x_1$, $W(\infty) = x_2$, and $W(0^+) = x_1^\sigma$. From Lemma 2.1, after a time-change, $W(\beta_0(t))$, $0 \le t < \infty$, has the same distribution as $\beta(t)$, $0 \le t < T$. Thus, a.s. $\lim_{t \to T^-} \beta(t) = W(\infty) = x_2$. □

3 Intermediate SLE($\kappa;\rho$) Process

Lemma 3.1 *For $\kappa\in(0,4)$ and $\rho\ge(\kappa-4)/2$, let $a=\frac{2\rho}{\kappa}$, $b=1-\frac{4}{\kappa}<0$, and $c=\frac{2\rho+4}{\kappa}\ge 1$. For $x\in(-1,1)$, let $U_0(x)={}_2F_1(a,b;c;x)$, where ${}_2F_1$ is the hypergeometric function* [6]. *Then there are $C_2>C_1>0$ such that $C_1\le U_0(x)\le C_2$ on $[0,1)$. Let $f_0(x)=\frac{U_0'(x)}{U_0(x)}$ on $[0,1)$. Then f_0 is also bounded on $[0,1)$, $f_0(x)\ge\frac{b}{1-x}$ for $0\le x<1$, and $\lim_{x\to1^-}f_0(x)=-\frac{a}{2}$.*

Proof It is known [6] that U_0 is analytic and satisfies the Gaussian hypergeometric equation:

$$x(x-1)U_0''(x)+[(a+b+1)x-c]U_0'(x)+abU_0(x)=0. \tag{3.1}$$

Moreover, we have $U_0(0)=1>0$ and $f_0'(0)=U_0'(0)=\frac{ab}{c}$. Let $z_0=\sup\{x\in(0,1):U_0(x)\ne0\}$. Then $z_0\in(0,1]$ and f_0 is analytic on $[0,z_0)$. Let $h_0(x)=f_0(x)-\frac{b}{1-x}=\frac{U_0'(x)}{U_0(x)}-\frac{b}{1-x}$ on $[0,z_0)$. Then $h_0(0)=\frac{ab}{c}-b=\frac{-4b}{2\rho+4}>0$. From (3.1) and that $b+c-a=1$, we find that for $x\in[0,z_0)$, $h_0(x)$ satisfies

$$xh_0'(x)+xh_0(x)^2+ch_0(x)+\frac{b(1-b)}{(1-x)^2}=0. \tag{3.2}$$

Assume that there is $x_1\in[0,z_0)$ such that $h_0(x_1)\le0$. Since $h_0(0)>0$, so $x_1>0$ and there is $x_0\in(0,z_0)$ such that $h_0(x_0)=0$ and $h_0(x)>0$ for $x\in[0,x_0)$. Then we have $h_0'(x_0)\le0$. However, since $b<0$, from (3.2) we have $h_0'(x_0)>0$, which is a contradiction. Thus $h_0(x)>0$ for all $x\in[0,z_0)$. So we have $f_0(x)>\frac{b}{1-x}$ for $0\le x<z_0$. Assume that $z_0<1$. Then z_0 is a zero of U_0, so z_0 is a simple pole of f_0, and the residue is positive. Thus, $\lim_{x\to z_0^-}f_0(x)=-\infty$, which contradicts that $f_0(x)>\frac{b}{1-x}$ for $0\le x<z_0$. Thus, $z_0=1$. So $U_0(x)\ne0$ and $f_0(x)>\frac{b}{1-x}$ for $0\le x<1$. Since $U_0(0)=1>0$, so $U_0(x)>0$ on $[0,1)$.

Now U_0 and f_0 are continuous on $[0,1)$, and $U_0(x)>0$ on $[0,1)$. To complete the proof, we suffice to show that $\lim_{x\to1^-}U_0(x)$ and $\lim_{x\to1^-}f_0(x)$ both exist and are finite, and $\lim_{x\to1^-}U_0(x)>0$. One may check that c, $c-a$, $c-b$ and $c-a-b$ are all positive. So from [6],

$$\lim_{x\to1^-}U_0(x)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\in(0,\infty). \tag{3.3}$$

We have $U_0'(x)=\frac{ab}{c}{}_2F_1(a+1,b+1;c+1;x)$. One may check that $c+1$ and $(c+1)-(a+1)-(b+1)$ are both positive. So from [6] again,

$$\lim_{x\to1^-}U_0'(x)=\frac{ab}{c}\cdot\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}. \tag{3.4}$$

From (3.3) and (3.4), we have $\lim_{x\to1^-}f_0(x)=\frac{ab}{c-a-b-1}=-\frac{a}{2}$, which is finite. □

From now on, fix $\kappa\in(0,4)$ and $\rho\ge(\kappa-4)/2$. Let f_0 be given by Lemma 3.1. Let

$$g_0(x):=\rho+\kappa xf_0(x). \tag{3.5}$$

From Lemma 3.1, g_0 is bounded on $[0,1)$, $\lim_{x\to1^-}g_0(x)=0$, and for $0\le x<1$,

$$g_0(x)\ge\rho+(\kappa-4)\frac{x}{1-x}. \tag{3.6}$$

For $0 < p_1 < p_2$, let

$$J(p_1, p_2) := -\left(\frac{1}{p_1} - \frac{1}{p_2}\right) g_0\left(\frac{p_1}{p_2}\right). \tag{3.7}$$

From (3.6) and that $\rho \geq \kappa/2 - 2$, we have

$$J(p_1, p_2) \leq \frac{\rho}{p_2} - \frac{\rho}{p_1} + \frac{4-\kappa}{p_2} \leq \frac{2-\kappa/2}{p_1} + \frac{2-\kappa/2}{p_2}. \tag{3.8}$$

Let $0 < p_1 < p_2$. Let $B(t)$ be a Brownian motion. Let $J(\cdot,\cdot)$ be defined by (3.7). Let $\xi(t)$, $p_1(t)$ and $p_2(t)$, $0 \leq t < T$, be the maximal solution to

$$\begin{cases} d\xi(t) = \sqrt{\kappa}\, dB(t) + J(p_1(t) - \xi(t), p_2(t) - \xi(t))\, dt, \\ dp_1(t) = \frac{2dt}{p_1(t) - \xi(t)}, \qquad dp_2(t) = \frac{2dt}{p_2(t) - \xi(t)}, \end{cases} \tag{3.9}$$

with initial values

$$\xi(0) = 0, \qquad p_j(0) = p_j, \quad j = 1, 2.$$

We call the chordal Loewner trace $\beta(t)$, $0 \leq t < T$, driven by ξ, a (generic) intermediate SLE$(\kappa;\rho)$ trace with force points p_1 and p_2. Note that $\xi(t) < p_1(t) < p_2(t)$ for $0 \leq t < T$. If $T < \infty$, we must have $\lim_{t\to T^-} p_1(t) - \xi(t) = 0$. Thus, if $\limsup_{t\to T^-} p_1(t) - \xi(t) > 0$, then $T = \infty$.

Theorem 3.1 *Let $\beta(t)$, $0 \leq t < T$, be an intermediate* SLE$(\kappa;\rho)$ *trace. Then a.s. $T = \infty$, which means that ∞ is a subsequential limit of $\beta(t)$ as $t \to T^-$.*

Proof Let $\xi(t)$, $0 \leq t < T$, be the driving function for β. Then there are $p_1(t)$, $p_2(t)$ and some Brownian motion $B(t)$ such that (3.9) holds, and $[0, T)$ is the maximal interval of the solution. Let $X_j(t) = p_j(t) - \xi(t)$, $j = 1, 2$. Then $0 < X_1(t) < X_2(t)$, $0 \leq t < T$; and for $j = 1, 2$, X_j satisfies the SDE

$$dX_j(t) = -\sqrt{\kappa}\, dB(t) + \left(\frac{2}{X_j(t)} - J(X_1(t), X_2(t))\right) dt.$$

From Itô's formula ([7]), for $j = 1, 2$, we have

$$d\ln(X_j(t)) = -\frac{\sqrt{\kappa}}{X_j(t)}\, dB(t) + \left(\frac{2-\kappa/2}{X_j(t)^2} - \frac{J(X_1(t), X_2(t))}{X_j(t)}\right) dt. \tag{3.10}$$

Thus, we have

$$\begin{aligned} d(\ln(X_2(t)/X_1(t))) = {} & \left(\frac{\sqrt{\kappa}}{X_1(t)} - \frac{\sqrt{\kappa}}{X_2(t)}\right) dB(t) - \left(\frac{2-\kappa/2}{X_1(t)^2} - \frac{2-\kappa/2}{X_2(t)^2}\right) dt \\ & + \left(\frac{1}{X_1(t)} - \frac{1}{X_2(t)}\right) J(X_1(t), X_2(t))\, dt. \end{aligned}$$

Since $1/X_1(t) > 1/X_2(t)$ and $2 - \kappa/2 > 0$, so from (3.8), the drift term for $\ln(X_2(t)/X_1(t))$ is not positive. Note that $\ln(X_2(t)/X_1(t))$ is always positive. So $(\ln(X_2(t)/X_1(t)))$ is a supermartingale. Thus, a.s. $\lim_{t\to T^-} \ln(X_2(t)/X_1(t))$ exists and is finite. So a.s.

$$\int_0^T \left(\frac{\sqrt{\kappa}}{X_1(t)} - \frac{\sqrt{\kappa}}{X_2(t)}\right)^2 dt = \lim_{t\to T^-} \langle \ln(X_2/X_1) \rangle_t < \infty. \tag{3.11}$$

Let $\mathcal{E}_1$ denote the event that $\lim_{t\to T^-}\ln(X_2(t)/X_1(t))>0$. Assume that $\mathcal{E}_1$ occurs. From (3.11), we have a.s. $\int_0^T X_1(t)^{-2}\,dt<\infty$. From (3.7) and (3.10), we have

$$d\ln(X_1(t))=-\frac{\sqrt{\kappa}}{X_1(t)}\,dB(t)+\frac{1}{X_1(t)^2}\left[2-\frac{\kappa}{2}+\left(1-\frac{X_1(t)}{X_2(t)}\right)g_0\left(\frac{X_1(t)}{X_2(t)}\right)\right]dt. \quad (3.12)$$

Since a.s. $\int_0^T X_1(t)^{-2}\,dt<\infty$, and g_0 is bounded on [0, 1), so a.s.

$$\int_0^T\frac{1}{X_1(t)^2}\left|2-\frac{\kappa}{2}+\left(1-\frac{X_1(t)}{X_2(t)}\right)g_0\left(\frac{X_1(t)}{X_2(t)}\right)\right|dt<\infty.$$

From (3.12) we have a.s. $\lim_{t\to T^-}\ln(X_1(t))$ exists and is finite. Thus, on $\mathcal{E}_1$ a.s. $\lim_{t\to T^-}X_1(t)$ exists and is positive, which implies that $T=\infty$.

Let $\mathcal{E}_2$ denote the event that $\lim_{t\to T^-}\ln(X_2(t)/X_1(t))=0$. Assume that $\mathcal{E}_1$ occurs. Then $\lim_{t\to T^-}X_1(t)/X_2(t)=1$, so $\lim_{t\to T^-}g_0(X_1(t)/X_2(t))=\lim_{x\to 1^-}g_0(x)=0$. Since $2-\kappa/2>0$, so the drift term in (3.12) is positive when t is close to T. From (3.12), a.s. $\limsup_{t\to T^-}\ln(X_1(t))>-\infty$, which implies that $\limsup_{t\to T^-}X_1(t)>0$. So we have a.s. $T=\infty$ on the event $\mathcal{E}_2$.

Since $\mathcal{E}_1\cup\mathcal{E}_2$ is a.s. the whole probability space, so a.s. $T=\infty$. Suppose $T=\infty$. Since for any $0<t<\infty$, the half-plane capacity of $\beta((0,t])$ is $2t$, so the diameter of $\beta((0,t])$ is at least $\sqrt{2t}$. Thus, the diameter of $\beta((0,\infty))$ is infinite, so ∞ is a subsequential limit of $\beta(t)$ as $t\to T^-$. □

The above theorem still holds if the force points p_1 and p_2 are random points, and the joint distribution of p_1 and p_2 is independent of the Brownian motion $B(t)$. The argument in the above proof still works.

We may let the force point p_1 be 0^+, and define the degenerate intermediate SLE($\kappa;\rho$) trace. The definition is as follows. Fix $p_2>0$. Let $\xi(t)$, $p_1(t)$ and $p_2(t)$ solve (3.9) for $0<t<T$, with initial values

$$\xi(0)=p_1(0)=0,\qquad p_2(0)=p_2. \quad (3.13)$$

Moreover, we require that

$$\xi(t)<p_1(t),\qquad 0<t<T. \quad (3.14)$$

The chordal Loewner trace $\beta(t)$, $0\le t<T$, driven by ξ, is called a degenerate intermediate SLE($\kappa;\rho$) trace with force points 0^+ and p_2.

We claim that the solution to (3.9) together with (3.13) and (3.14) a.s. exists. For the proof, we suffice to prove that the solution exists on $(0,T_0)$ for some stopping time $T_0>0$ because after T_0 we are dealing with some generic case with random force points. Let $\widetilde{B}(t)$ be a Brownian motion under some probability measure $\mathbf{P}$. Let $\xi(t)$, $p_1(t)$ and $p_2(t)$, $0<t<T_1$, be the maximal solution to

$$\begin{cases} d\xi(t)=\sqrt{\kappa}\,d\widetilde{B}(t)+\frac{\rho}{\xi(t)-p_1(t)}\,dt,\\ dp_j(t)=\frac{2dt}{p_j(t)-\xi(t)},\qquad j=1,2,\end{cases}$$

such that (3.13) and (3.14) hold. The solution a.s. exists because ξ is the driving function for an SLE($\kappa;\rho$) process started from $(0,0^+)$.

From (3.5) and (3.7), it is clear that $\lim_{p_1\to 0^+}(J(p_1,p_2)+\frac{\rho}{p_1})=\frac{\rho}{p_2}-\frac{\kappa}{p_2}f_0(0)$. Define $Z(t)$, $0\le t<T_1$, such that for $t>0$, $Z(t)=J(p_1(t)-\xi(t),p_2(t)-\xi(t))-\frac{\rho}{\xi(t)-p_1(t)}$, and

$Z(0)=\frac{\rho}{p_2}-\frac{\kappa}{p_2}f_0(0)$. Then $Z(t)$ is continuous on $[0,T_1)$. From the Girsanov's Theorem, there is a stopping time $T_0\in(0,T_1)$ such that under some other probability measure $\mathbf{Q}$, $B(t):=\widetilde{B}(t)-\frac{1}{\sqrt{\kappa}}\int_0^t Z(s)\,ds$, $0\le t<T_0$, is a partial Brownian motion, which means that $B(t)$ could be extended to a full Brownian motion. Then we have

$$d\xi(t)=\sqrt{\kappa}\,dB(t)+J(p_1(t)-\xi(t),p_2(t)-\xi(t))\,dt,\quad 0\le t<T_0.$$

Thus, the solution to (3.9) with (3.13) and (3.14) a.s. exists on $(0,T_0)$. Then the solution can be extended to the maximal interval, say $(0,T)$, and so we have the existence of the maximal solution. From Theorem 3.1, we get the following corollary.

Corollary 3.1 *Let $\beta(t)$, $0\le t<T$, be a degenerate intermediate* SLE$(\kappa;\rho)$ *trace. Then a.s. $T=\infty$, which means that ∞ is a subsequential limit of $\beta(t)$ as $t\to T^-$.*

4 Martingales

Fix $\kappa\in(0,4)$ and $\rho\ge\kappa/2-2$. Let $x_1<x_2\in\mathbb{R}$, $\sigma_1=+$ and $\sigma_2=-$. Throughout this section, the subscripts j and k will be any of the two numbers: 1 or 2, such that j and k are different. Let $\xi_j(t)$, $0\le t<T_j$, be the driving function for a chordal SLE$(\kappa;\rho,\kappa-6-\rho)$ trace $\beta_j(t)$, $0\le t<T_j$, started from $(x_j;x_j^{\sigma_j},x_k)$. From Lemma 2.3, we have a.s. $\lim_{t\to T_j^-}\beta_j(t)=x_k$. Let $\varphi_j(t,\cdot)$ and $K_j(t)$, $0\le t<T_j$, be the chordal Loewner maps and hulls driven by ξ_j. Let $p_j(t)$ and $q_j(t)$ be the force point functions started from $x_j^{\sigma_j}$ and x_k, respectively. So we have $p_j(t)=\lim_{x\to x_j^{\sigma_j}}\varphi_j(t,x)$ and $q_j(t)=\varphi_j(t,x_k)$. For $0\le t<T$, let

$$B_j(t)=\frac{1}{\sqrt{\kappa}}\left(\xi_j(t)-x_j-\int_0^t\frac{\rho}{\xi_j(s)-p_j(s)}\,ds+\int_0^t\frac{\kappa-6-\rho}{\xi_j(s)-q_j(s)}\,ds\right).$$

Then $B_j(t)$, $0\le t<T$, is a partial Brownian motions. Let $(\mathcal{F}_t^j)$ be the filtration generated by $B_j(t)$. Then $(\xi_j(t))$, $p_j(t)$, and $(q_j(t))$ are all $(\mathcal{F}_t)$-adapted. And $(\xi_j(t))$ is an $(\mathcal{F}_t)$-semi-martingale with $d\langle\xi\rangle_t=\kappa\,dt$. Moreover, $\xi_j(t)$, $p_j(t)$ and $q_j(t)$, $0<t<T_j$, are the maximal solution to the following equations

$$d\xi_j(t)=\sqrt{\kappa}\,dB_j(t)+\frac{\rho}{\xi_j(t)-p_j(t)}\,dt+\frac{\kappa-6-\rho}{\xi_j(t)-q_j(t)}\,dt,\tag{4.1}$$

$$dp_j(t)=\frac{2}{p_j(t)-\xi_j(t)}\,dt,\tag{4.2}$$

$$dq_j(t)=\frac{2}{q_j(t)-\xi_j(t)}\,dt,\tag{4.3}$$

with initial values

$$\xi_j(0)=p_j(0)=x_j,\qquad q_j(0)=x_k;\tag{4.4}$$

and they satisfy the inequalities

$$\xi_1(t)<p_1(t)<q_1(t),\quad 0<t<T_1;\qquad \xi_2(t)>p_2(t)>q_2(t),\quad 0<t<T_2.\tag{4.5}$$

Now suppose that $(\xi_1(t))$ and $(\xi_2(t))$ are independent. Then $(B_1(t))$ and $(B_2(t))$ are also independent. So for any fixed $(\mathcal{F}^k_t)$-stopping time t_k with $0\le t_k<T_k$, $B_j(t)$, $0\le t<T_j$, is a partial $(\mathcal{F}^j_t\times\mathcal{F}^k_{t_k})_{t\ge 0}$-Brownian motion.

Differentiating (2.1) w.r.t. ∂_z and plugging $\xi=\xi_j$ and $z=x_k$, we find that for $0\le t<T_j$,

$$\frac{d\partial_z\varphi_j(t,x_k)}{\partial_z\varphi_j(t,x_k)}=\frac{-2dt}{(q_j(t_j)-\xi_j(t_j))^2}. \tag{4.6}$$

From (4.1)–(4.3) we have that, for $0<t<T_j$,

$$\frac{d(\xi_j(t)-p_j(t))}{\xi_j(t)-p_j(t)}=\frac{d\xi_j(t)}{\xi_j(t)-p_j(t)}+\frac{2dt}{(\xi_j(t)-p_j(t))^2}; \tag{4.7}$$

$$\frac{d(\xi_j(t)-q_j(t))}{\xi_j(t)-q_j(t)}=\frac{d\xi_j(t)}{\xi_j(t)-q_j(t)}+\frac{2dt}{(\xi_j(t)-q_j(t))^2}; \tag{4.8}$$

$$\frac{d(q_j(t)-p_j(t))}{q_j(t)-p_j(t)}=\frac{-2dt}{(\xi_j(t)-q_j(t))(\xi_j(t)-p_j(t))}. \tag{4.9}$$

In the above equations, (4.6) and (4.9) are ODEs, (4.7) and (4.8) are $(\mathcal{F}^j_t)$-adapted SDEs.

For $t\in(0,T_j)$, define

$$r_j(t)=|\xi_j(t)-p_j(t)|^{-\frac{\rho}{\kappa}}|\xi_j(t)-q_j(t)|^{-\frac{\kappa-6-\rho}{\kappa}}|q_j(t)-p_j(t)|^{-\frac{\rho(\kappa-6-\rho)}{2\kappa}}\partial_z\varphi_j(t,x_k)^{\frac{(\rho+2)(\kappa-6-\rho)}{4\kappa}}. \tag{4.10}$$

From (4.1), (4.6)–(4.9) and Itô's formula, we have that, for $t>0$,

$$\frac{dr_j(t)}{r_j(t)}=-\frac{\rho}{\xi_j(t)-p_j(t)}\cdot\frac{dB_j(t)}{\sqrt{\kappa}}-\frac{\kappa-6-\rho}{\xi_j(t)-q_j(t)}\cdot\frac{dB_j(t)}{\sqrt{\kappa}}+\frac{\rho(\kappa-4-\rho)/(2\kappa)}{(\xi_j(t)-p_j(t))^2}\,dt. \tag{4.11}$$

Let $\mathcal{D}=\{(t_1,t_2)\in[0,T_1)\times[0,T_2):\beta_1([0,t_1])\cap\beta_2([0,t_2])=\emptyset\}$. Then for any $(t_1,t_2)\in\mathcal{D}$, $K_1(t_1)\cup K_2(t_2)$ is a hull in $\mathbb{H}$ w.r.t. ∞. For $(t_1,t_2)\in\mathcal{D}$, let

$$K_{k,t_j}(t_k):=(K_j(t_j)\cup K_k(t_k))/K_j(t_j)=\varphi_j(t_j,K_k(t_k)), \tag{4.12}$$

and $\varphi_{k,t_j}(t_k,\cdot):=\varphi_{K_{k,t_j}(t_k)}$. Then $K_{k,t_j}(t_k)$ is the image of a curve in $\mathbb{H}$ started from $\varphi_j(t_j,x_k)=q_j(t_j)$. And for any $z\in\mathbb{H}\setminus(K_1(t_1)\cup K_2(t_2))$,

$$\varphi_{K_1(t_1)\cup K_2(t_2)}(z)=\varphi_{1,t_2}(t_1,\varphi_2(t_2,z))=\varphi_{2,t_1}(t_2,\varphi_1(t_1,z)). \tag{4.13}$$

Define $A_{j,h}$, $h\in\mathbb{Z}_{\ge 0}$, on $\mathcal{D}$ such that $A_{j,h}(t_1,t_2)=\partial_z^h\varphi_{k,t_j}(t_k,\xi_j(t_j))$. Note that the definition of $A_{j,h}$ here agrees with the definition of $A_{j,h}$ in Sect. 4.2 of [10]. From now on, we fix t_k to be some $(\mathcal{F}^k_t)$-stopping time that lies on $[0,T_k)$, and consider the filtration $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$. Since $B_j(t)$ and $B_k(t)$ are independent Brownian motions, so $B_j(t_j)$ is an $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-Brownian motion. We use ∂_j to denote the partial derivative w.r.t. t_j. The following equations are (4.10) and (4.12) in [10], where (4.14) is an $(\mathcal{F}^j_{t_j}\times\mathcal{F}^k_{t_k})_{t_j\ge 0}$-adapted SDE.

$$\partial_jA_{j,0}=A_{j,1}\partial\xi_j(t_j)+\left(\frac{\kappa}{2}-3\right)A_{j,2}\partial t_j; \tag{4.14}$$

$$\partial_jA_{k,0}=\frac{2A_{j,1}^2}{A_{k,0}-A_{j,0}},\qquad \frac{\partial_jA_{k,1}}{A_{k,1}}=\frac{-2A_{j,1}^2}{(A_{k,0}-A_{j,0})^2}. \tag{4.15}$$

We now use ∂_1 and ∂_z to denote the partial derivatives of $\varphi_{j,t_0}(\cdot,\cdot)$ w.r.t. the first (real) and second (complex) variables, respectively, inside the bracket; and use ∂_0 to denote the partial derivative of $\varphi_{j,t_0}(\cdot,\cdot)$ w.r.t. the subscript t_0. Let $(t_1,t_2)\in\mathcal{D}$. The following equations are (3.9) and (3.15) in [9].

$$\partial_1\varphi_{j,t_k}(t_j,z)=\frac{2A_{j,1}^2}{\varphi_{j,t_k}(t_j,z)-A_{j,0}},\quad z\in\mathbb{H}\setminus K_{j,t_k}(t_j); \tag{4.16}$$

$$\partial_0\varphi_{k,t_j}(t_k,z)=\frac{2A_{j,1}^2}{\varphi_{k,t_j}(t_k,z)-A_{j,0}}-\frac{2\partial_z\varphi_{k,t_j}(t_k,z)}{z-\xi_j(t_j)},\quad z\in\mathbb{H}\setminus K_{k,t_j}(t_k). \tag{4.17}$$

Since $\overline{K_{j,t_k}(t_j)}\cap\mathbb{R}=\{q_k(t_k)\}$ and $\overline{K_{k,t_j}(t_k)}\cap\mathbb{R}=\{q_j(t_j)\}$, so after continuation, (4.16) also holds for any $z\in\mathbb{R}\setminus\{q_k(t_k)\}$, and (4.17) also holds for any $z\in\mathbb{R}\setminus\{\xi_j(t_j),q_j(t_j)\}$. Differentiating (4.17) w.r.t. ∂_z, we find that for $(t_1,t_2)\in\mathcal{D}$, and $z\in\mathbb{R}\setminus\{\xi_j(t_j),q_j(t_j)\}$,

$$\partial_0\partial_z\varphi_{k,t_j}(t_k,z)=-\frac{2A_{j,1}^2\partial_z\varphi_{k,t_j}(t_k,z)}{(\varphi_{k,t_j}(t_k,z)-A_{j,0})^2}-\frac{2\partial_z^2\varphi_{k,t_j}(t_k,z)}{z-\xi_j(t_j)}+\frac{2\partial_z\varphi_{k,t_j}(t_k,z)}{(z-\xi_j(t_j))^2}. \tag{4.18}$$

Define $B_{j,0}$ on $\mathcal{D}$ such that $B_{j,0}(t_1,t_2)=\varphi_{k,t_j}(t_k,p_j(t_j))$. Since $\xi_1(0)=p_1(0)$ and $\xi_1(t)<p_1(t)$ for $t>0$, so $A_{1,0}(0,t_2)=B_{1,0}(0,t_2)$ and $A_{1,0}(t_1,t_2)<B_{1,0}(t_1,t_2)$ if $t_1>0$. Similarly, we have $A_{2,0}(t_1,0)=B_{2,0}(t_1,0)$ and $A_{2,0}(t_1,t_2)>B_{2,0}(t_1,t_2)$ if $t_2>0$. Choose any $y_1<y_2\in(x_1,x_2)$. Then $p_1(t_1)\le\varphi_1(t_1,y_1)<\varphi_2(t_1,y_2)$ for any $t_1\in[0,T_1)$. From (4.13) we have

$$B_{1,0}(t_1,t_2)\le\varphi_{K_1(t_1)\cup K_2(t_2)}(y_1)<\varphi_{K_1(t_1)\cup K_2(t_2)}(y_2)$$

for any $(t_1,t_2)\in\mathcal{D}$. Similarly, $B_{2,0}(t_1,t_2)\ge\varphi_{K_1(t_1)\cup K_2(t_2)}(y_2)>\varphi_{K_1(t_1)\cup K_2(t_2)}(y_1)$ for any $(t_1,t_2)\in\mathcal{D}$. Thus, $B_{1,0}<B_{2,0}$ on $\mathcal{D}$. So in general, $A_{1,0}\le B_{1,0}<B_{2,0}\le A_{2,0}$, where $A_{1,0}=B_{1,0}$ iff $t_1=0$, and $B_{2,0}=A_{2,0}$ iff $t_2=0$.

Let $(t_1,t_2)\in\mathcal{D}$. Since $p_k(t_k)\ne q_k(t_k)$, so we may apply (4.16) with $z=p_k(t_k)$, and obtain

$$\partial_j B_{k,0}=\frac{2A_{j,1}^2}{B_{k,0}-A_{j,0}}. \tag{4.19}$$

Now suppose $t_j>0$. Then $p_j(t_j)\in\mathbb{R}\setminus\{\xi_j(t_j),q_j(t_j)\}$. So we may apply (4.17) with $z=p_j(t_j)$, and use (4.2) and chain rule to obtain

$$\partial_j B_{j,0}=\frac{2A_{j,1}^2}{B_{j,0}-A_{j,0}}. \tag{4.20}$$

Note that (4.19) and (4.20) have the same forms as the formula for $\partial_j B_{m,0}$ in (4.13) in [10]. But here we require that $t_j>0$ in (4.20).

Let $E_{j,0}=A_{j,0}-A_{k,0}=-E_{k,0}\ne0$, $E_{j,m}=A_{j,0}-B_{m,0}$, $m=1,2$, and $C_{j,k}=B_{j,0}-B_{k,0}=-C_{k,j}\ne0$. From (4.14)–(4.15) and (4.19)–(4.20), we obtain the following formulas, which have the same forms as (4.14) and (4.15) in [10].

$$\frac{\partial_j E_{j,m}}{E_{j,m}}=\frac{A_{j,1}}{E_{j,m}}\partial\xi_j(t_j)+\left(\left(\frac{\kappa}{2}-3\right)\cdot\frac{A_{j,2}}{E_{j,m}}+2\cdot\frac{A_{j,1}^2}{E_{j,m}^2}\right)\partial t_j,\quad m=0,1,2; \tag{4.21}$$

$$\frac{\partial_j E_{k,m}}{E_{k,m}}=\frac{-2A_{j,1}^2}{E_{j,0}E_{j,m}}\partial t_j,\quad m=1,2; \tag{4.22}$$

$$\frac{\partial_j C_{j,k}}{C_{j,k}} = \frac{-2A_{j,1}^2}{E_{j,1}E_{j,2}}\,\partial t_j. \tag{4.23}$$

Here we require that $t_j > 0$ in the SDEs for $\partial_j E_{j,j}$, $\partial_j E_{k,j}$, and $\partial_j C_{j,k}$, because (4.20) does not hold for $t_j = 0$.

Define $\widetilde{B}_{j,1}$ on $\mathcal{D}$ such that $\widetilde{B}_{j,1}(t_1,t_2) = \partial_z \varphi_{k,t_j}(t_k, p_j(t_j))$. Differentiating (4.16) w.r.t. ∂_z and plugging $z = p_k(t_k)$, we get

$$\frac{\partial_j \widetilde{B}_{k,1}}{\widetilde{B}_{k,1}} = \frac{-2A_{j,1}^2}{E_{j,k}^2}\,\partial t_j. \tag{4.24}$$

Applying (4.18) with $z = p_j(t_j)$, and using (4.2) and chain rule, we find that, for $t_j > 0$,

$$\frac{\partial_j \widetilde{B}_{j,1}}{\widetilde{B}_{j,1}} = \frac{-2A_{j,1}^2}{E_{j,j}^2}\,\partial t_j + \frac{2}{(p_j(t_j) - \xi_j(t_j))^2}\,\partial t_j. \tag{4.25}$$

Let $D = \frac{\widetilde{B}_{1,1}\widetilde{B}_{2,1}}{C_{1,2}^2} = \frac{\widetilde{B}_{1,1}\widetilde{B}_{2,1}}{C_{2,1}^2}$. From (4.23)–(4.25), we find that, for $t_j > 0$,

$$\frac{\partial_j D}{D} = -2\left(\frac{A_{1,1}}{E_{j,j}} - \frac{A_{1,1}}{E_{j,k}}\right)^2 \partial t_j + \frac{2}{(p_j(t_j) - \xi_j(t_j))^2}\,\partial t_j. \tag{4.26}$$

Let $\mathcal{D}' = \{(t_1,t_2) \in \mathcal{D} : t_1 * t_2 \neq 0\}$. Define R on $\mathcal{D}$ such that $R = \frac{E_{1,1}E_{2,2}}{E_{1,2}E_{2,1}} = \frac{E_{j,j}E_{k,k}}{E_{j,k}E_{k,j}}$. From $A_{1,0} \le B_{1,0} < B_{2,0} \le A_{2,0}$ we have $|E_{j,j}| < |E_{j,k}|$ and $E_{j,j}/E_{j,k} \ge 0$, so $R \in [0,1)$. Since $A_{j,0} \neq B_{j,0}$ when $t_j > 0$, so $E_{1,1} * E_{2,2} \neq 0$ on $\mathcal{D}'$. Thus, $R \in (0,1)$ on $\mathcal{D}'$. Since $E_{k,m} = E_{j,m} - E_{j,0}$ for $m = 1,2$, so we have

$$\frac{R+1}{R-1} = \frac{2/E_{j,0}}{1/E_{j,j} - 1/E_{j,k}} - \frac{1/E_{j,j} + 1/E_{j,k}}{1/E_{j,j} - 1/E_{j,k}}. \tag{4.27}$$

From (4.21) and (4.22), we have that, for $t_j > 0$,

$$\begin{aligned}\partial_j R = {} & R\left(\frac{A_{j,1}}{E_{j,j}} - \frac{A_{j,1}}{E_{j,k}}\right)\partial \xi_j(t_j) + R\left[\left(\frac{\kappa}{2} - 3\right)\left(\frac{A_{j,2}}{E_{j,j}} - \frac{A_{j,2}}{E_{j,k}}\right) + \frac{\kappa}{2}\left(\frac{A_{j,1}}{E_{j,j}} - \frac{A_{j,1}}{E_{j,k}}\right)^2\right. \\ & \left. + \left(2 - \frac{\kappa}{2}\right)\left(\frac{A_{j,1}^2}{E_{j,j}^2} - \frac{A_{j,1}^2}{E_{j,k}^2}\right) + \left(\frac{2A_{j,1}^2}{E_{j,0}E_{j,j}} - \frac{2A_{j,1}^2}{E_{j,0}E_{j,k}}\right)\right]\partial t_j. \end{aligned} \tag{4.28}$$

Let $U_0(x)$ and $f_0(x)$ be given by Lemma 3.1. Let g_0 be defined by (3.5). For $x \in (0,1)$, let $V_0(x) := x^{\frac{\rho}{\kappa}} U_0(x)$. From (3.1) and (3.5), we find that $V_0(x)$ satisfies

$$x\frac{V_0'(x)}{V_0(x)} = \frac{g_0(x)}{\kappa}, \tag{4.29}$$

$$\frac{\kappa}{2}\,\frac{V_0''(x)}{V_0(x)}x^2 = \left[\left(2 - \frac{\kappa}{2}\right)\frac{x+1}{x-1} - \frac{\kappa}{2}\right]\frac{g_0(x)}{\kappa} - \frac{\rho(\kappa - 4 - \rho)}{2\kappa}. \tag{4.30}$$

Since $R \in (0,1)$ on $\mathcal{D}'$, so $V_0(R)$ is well defined on $\mathcal{D}'$. From (4.27)–(4.30), we have that

$$\frac{\partial_j V_0(R)}{V_0(R)} = \frac{g_0(R)}{\kappa}\left(\frac{A_{j,1}}{E_{j,j}} - \frac{A_{j,1}}{E_{j,k}}\right)\partial \xi_j(t_j) + \frac{g_0(R)}{\kappa}\left(\frac{\kappa}{2} - 3\right)\left[\left(\frac{A_{j,2}}{E_{j,j}} - \frac{A_{j,2}}{E_{j,k}}\right)\right.$$

$$- \left(\frac{2A_{j,1}^2}{E_{j,0}E_{j,j}} - \frac{2A_{j,1}^2}{E_{j,0}E_{j,k}} \right) \bigg] \partial t_j$$
$$- \frac{\rho(\kappa-4-\rho)}{2\kappa} \left(\frac{A_{j,1}}{E_{j,k}} - \frac{A_{j,1}}{E_{j,j}} \right)^2 \partial t_j. \tag{4.31}$$

Define N and F on $\mathcal{D}$ such that $N = \frac{A_{1,1}A_{2,1}}{(A_{1,0}-A_{2,0})^2}$ and $F(t_1,t_2) = \exp(\int_0^{t_2}\int_0^{t_1} 2N(s_1,s_2)\,ds_1\,ds_2)$. Let $\alpha = \frac{6-\kappa}{2\kappa}$ and $\lambda = \frac{(8-3\kappa)(6-\kappa)}{2\kappa}$. The following equations are (4.13) in [9] and (4.25) in [10].

$$\frac{\partial_j N^\alpha}{N^\alpha} = \frac{1}{\kappa}\left(3 - \frac{\kappa}{2}\right)\left(\frac{A_{j,2}}{A_{j,1}} - \frac{2A_{j,1}}{E_{j,0}}\right)\partial \xi_j(t_j) + \lambda\left(\frac{1}{4}\cdot\frac{A_{1,2}^2}{A_{1,1}^2} - \frac{1}{6}\cdot\frac{A_{1,3}}{A_{1,1}}\right)\partial t_j; \tag{4.32}$$

$$\frac{\partial_j F^{-\lambda}}{F^{-\lambda}} = -\lambda\left(\frac{1}{4}\cdot\frac{A_{j,2}^2}{A_{j,1}^2} - \frac{1}{6}\cdot\frac{A_{j,3}}{A_{j,1}}\right)\partial t_j. \tag{4.33}$$

Let $\tau = \frac{(\rho+2)(\kappa-6-\rho)}{2\kappa}$ and $\delta = -\frac{\rho(\kappa-4-\rho)}{4\kappa}$. Define M on $\mathcal{D}'$ such that

$$M = |x_1 - x_2|^\tau r_1(t_1) r_2(t_2) D^\delta V_0(R) N^\alpha F^{-\lambda}. \tag{4.34}$$

From (4.1), (4.11), (4.26) and (4.31)–(4.33), we get

$$\frac{\partial_j M}{M} = \bigg[\left(3 - \frac{\kappa}{2}\right)\left(\frac{A_{j,2}}{A_{j,1}} - \frac{2A_{j,1}}{E_{j,0}}\right) + g_0(R)\left(\frac{A_{j,1}}{E_{j,j}} - \frac{A_{j,1}}{E_{j,k}}\right)$$
$$- \frac{\rho}{\xi_j(t_j) - p_j(t_j)} - \frac{\kappa-6-\rho}{\xi_j(t_j) - q_j(t_j)}\bigg] \frac{dB_j(t_j)}{\sqrt{\kappa}}. \tag{4.35}$$

Define $\widetilde{r}_j$ on $[0, T_j)$ such that

$$\widetilde{r}_j(t_j) = |\xi_j(t_j) - q_j(t_j)|^{-\frac{\kappa-6-\rho}{\kappa}} |q_j(t_j) - p_j(t_j)|^{-\frac{\rho(\kappa-6-\rho)}{2\kappa}} \partial_z \varphi_j(t_j, x_{3-j})^{\frac{(\rho+2)(\kappa-6-\rho)}{4\kappa}}. \tag{4.36}$$

Define $\widetilde{M}$ on $\mathcal{D}$ such that

$$\widetilde{M} = |x_1 - x_2|^\tau \widetilde{r}_1(t_1)\widetilde{r}_2(t_2) D^\delta |E_{1,2}E_{2,1}|^{-\frac{\rho}{\kappa}} U_0(R) N^\alpha F^{-\lambda}. \tag{4.37}$$

Then $\widetilde{M}$ is continuous on $\mathcal{D}$. Define L_j on $\mathcal{D}$ such that if $t_j = 0$ then $L_j = \partial_z \varphi_k(t_k, x_j)$; if $t_j > 0$ then

$$L_j(t_1, t_2) = \frac{|E_{j,j}(t_1,t_2)|}{|\xi_j(t_j) - p_j(t_j)|} = \frac{\varphi_{k,t_j}(t_k, \xi_j(t_j)) - \varphi_{k,t_j}(t_k, p_j(t_j))}{\xi_j(t_j) - p_j(t_j)}. \tag{4.38}$$

Here the second "=" holds because $E_{j,j}$ has the same sign as $\xi_j(t_j) - p_j(t_j)$. Since $\lim_{t_k \to 0^+} \xi_k(t_k) = \lim_{t_k \to 0^+} p_k(t_k) = x_k$ and $\lim_{t_j \to 0^+} \varphi_{k,t_j}(t_k, \cdot) = \varphi_{k,0}(t_k, \cdot) = \varphi_k(t_k, \cdot)$, so L_j is continuous on $\mathcal{D}$. From (4.10), (4.34), (4.36)–(4.38), and that $V_0(x) = x^{\frac{\rho}{\kappa}} U_0(x)$, we find that $M = \widetilde{M} L_1^{\frac{\rho}{\kappa}} L_2^{\frac{\rho}{\kappa}}$ on $\mathcal{D}'$. Thus M has continuous extension to $\mathcal{D}$. Now we check the value of M when $t_j = 0$.

We have $\xi_j(0) = p_j(0) = x_j$, $q_j(0) = x_k$, and $K_j(0) = \emptyset$. So $K_j(0) \cup K_k(t_k) = K_k(t_k)$. From (4.12) we have $K_{k,0}(t_k) = K_k(t_k)$ and $K_{j,t_k}(0) = \emptyset$, which implies that $\varphi_{k,0}(t_k, \cdot) = \varphi_k(t_k, \cdot)$ and $\varphi_{j,t_k}(0, \cdot) = \mathrm{id}$. Thus, if $t_j = 0$, then $\widetilde{r}_j(t_j) = |x_j - x_k|^{-\tau}$; and $A_{j,0} =$

$\varphi_k(t_k,x_j)=q_k(t_k)=B_{j,0}$, $A_{j,1}=\partial_z\varphi_k(t_k,x_j)=\widetilde{B}_{j,1}$, $A_{j,2}=\partial_z^2\varphi_k(t_k,x_j)$, $A_{k,0}=\xi_k(t_k)$, $B_{k,0}=p_k(t_k)$, and $A_{k,1}=1=\widetilde{B}_{k,1}$, which imply that $E_{j,j}=0$, $E_{j,k}=q_k(t_k)-p_k(t_k)$, $E_{k,0}=E_{k,j}=\xi_k(t_k)-q_k(t_k)=-E_{j,0}$, $E_{k,k}=\xi_k(t_k)-p_k(t_k)$, $|C_{j,k}|=|p_k(t_k)-q_k(t_k)|$, $D=\frac{\partial_z\varphi_k(t_k,x_j)}{(p_k(t_k)-q_k(t_k))^2}$, $R=0$, $U_0(R)=1$, $N=\frac{\partial_z\varphi_k(t_k,x_j)}{(\xi_k(t_k)-q_k(t_k))^2}$, and $F=1$. From (4.36), (4.37) and the above argument, we find that $\widetilde{M}=\partial_z\varphi_k(t_k,x_j)^{-\frac{\rho}{\kappa}}$ when $t_j=0$. From the definition, $L_j=\partial_z\varphi_k(t_k,x_j)$ when $t_j=0$. Since $\varphi_{j,t_k}(0,\cdot)=\text{id}$, so $L_k=1$ when $t_j=0$. Thus, after continuous extension, $M=1$ when t_1 or t_2 equals 0.

Let Q_j be the formula inside the square bracket in (4.35), that is,

$$Q_j=\left(3-\frac{\kappa}{2}\right)\left(\frac{A_{j,2}}{A_{j,1}}-\frac{2A_{j,1}}{E_{j,0}}\right)+g_0(R)\left(\frac{A_{j,1}}{E_{j,j}}-\frac{A_{j,1}}{E_{j,k}}\right)\\ -\frac{\rho}{\xi_j(t_j)-p_j(t_j)}-\frac{\kappa-6-\rho}{\xi_j(t_j)-q_j(t_j)}. \tag{4.39}$$

Then Q_j is defined on $\mathcal{D}'$. Using the observation in the previous paragraph and the fact that $g_0(0)=\rho$ and g_0 is differentiable at 0, we may check that Q_j has continuous extension to $\mathcal{D}$. Thus, after continuous extensions, the formula $\frac{\partial_j M}{M}=Q_j\frac{dB_j(t_j)}{\sqrt{\kappa}}$ holds in $\mathcal{D}$. For each $t_k\in[0,T_k)$, let $T_j(t_k)$ be the maximal number such that $K_j(t)\cap K_k(t_k)=\emptyset$ for $0\le t<T_j(t_k)$. From (4.35) we conclude that for any fixed stopping time $t_k\in[0,T_k)$, M is a continuous local martingale in t_j, where t_j ranges in $[0,T_j)$.

Let HP denote the set of (H_1,H_2) such that for $j=1,2$, H_j is a hull in $\mathbb{H}$ w.r.t. ∞ that contains some neighborhood of x_j in $\mathbb{H}$, and $\overline{H_1}\cap\overline{H_2}=\emptyset$. For $(H_1,H_2)\in\text{HP}$, let $T_j(H_j)$ be the first t such that $\beta_j(t_j)\in\overline{\mathbb{H}\setminus H_j}$. Then $T_j(H_j)$ is an $(\mathcal{F}_t^j)$-stopping time.

Theorem 4.1 *For any $(H_1,H_2)\in\text{HP}$, there are $C_2>C_1>0$ depending on H_1 and H_2 such that $C_1\le M(t_1,t_2)\le C_2$ for $(t_1,t_2)\in[0,T_1(H_1)]\times[0,T_2(H_2)]$.*

Proof Since $M=\widetilde{M}L_1^{\frac{\rho}{\kappa}}L_2^{\frac{\rho}{\kappa}}$, so we suffice to show that the theorem holds for $\widetilde{M}$ and L_j, $j=1,2$. To check the boundedness of $\widetilde{M}$, we suffice to show that the theorem holds for every factor on the right-hand side of (4.37). From Lemma 3.1, we find that the theorem holds for $U_0(R)$. The boundedness of other factors in (4.37) can be proved using the method in Sect. 5 of [9]. For the boundedness of L_j, we suffice to note that from Lemma 5.2 in [9], the value of L_j lies between $A_{j,1}$ and $\widetilde{B}_{j,1}$, which are both uniformly bounded from ∞ and 0. □

Fix $(H_1,H_2)\in\text{HP}$. From the local martingale property of M and the above theorem, we see that $\mathbf{E}[M(T_1(H_1),T_2(H_2))]=1$. Let μ denote the joint distribution of $(\xi_1(t),0\le t<T_1)$ and $(\xi_2(t),0\le t<T_2)$. Define ν such that $d\nu/d\mu=M(T_1(H_1),T_2(H_2))$. Then ν is also a probability measure. Suppose temporarily that the joint distribution of ξ_1 and ξ_2 is ν instead of μ. For $(t_1,t_2)\in\mathcal{D}$, define

$$B_{1,t_2}(t_1)=B_1(t_1)-\frac{1}{\sqrt{\kappa}}\int_0^{t_1}Q_1(s,t_2)\,ds,\\ B_{2,t_1}(t_2)=B_2(t_2)-\frac{1}{\sqrt{\kappa}}\int_0^{t_2}Q_2(t_1,s)\,ds. \tag{4.40}$$

Fix an $(\mathcal{F}^k_t)$-stopping time $\bar{t}_k$ with $\bar{t}_k \le T_k(H_k)$. Since $B_j(t)$ is an $(\mathcal{F}^j_t \times \mathcal{F}^k_{\bar{t}_k})_{t\ge 0}$-Brownian motion under μ, so from (4.35), (4.39) and the Girsanov's Theorem, $B_{j,\bar{t}_k}(t)$, $0 \le t \le T_j(H_j)$, is a partial $(\mathcal{F}^j_t \times \mathcal{F}^k_{\bar{t}_k})_{t\ge 0}$-Brownian motion under ν.

The following theorem is Theorem 6.1 in [9] and Theorem 4.5 in [10]. It can be proved using the above theorem and the argument in [9] or [10].

Theorem 4.2 *For any* $(H^m_1, H^m_2) \in \mathrm{HP}$, $1 \le m \le n$, *there is a continuous function* $M_*(t_1,t_2)$ *defined on* $[0,\infty]^2$ *that satisfies the following properties*: (i) $M_* = M$ *on* $[0, T_1(H^m_1)] \times [0, T_2(H^m_2)]$ *for* $m = 1, \ldots, n$; (ii) $M_*(t,0) = M_*(0,t) = 1$ *for any* $t \ge 0$; (iii) $M_*(t_1,t_2) \in [C_1, C_2]$ *for any* $t_1, t_2 \ge 0$, *where* $C_2 > C_1 > 0$ *are constants depending only on* H^m_j, $j = 1, 2$, $1 \le m \le n$; (iv) *for any* $(\mathcal{F}^2_t)$*-stopping time* $\bar{t}_2$, $(M_*(t_1, \bar{t}_2), t_1 \ge 0)$ *is a bounded continuous* $(\mathcal{F}^1_{t_1} \times \mathcal{F}^2_{\bar{t}_2})_{t_1\ge 0}$*-martingale*; *and* (v) *for any* $(\mathcal{F}^1_t)$*-stopping time* $\bar{t}_1$, $(M_*(\bar{t}_1, t_2), t_2 \ge 0)$ *is a bounded continuous* $(\mathcal{F}^1_{\bar{t}_1} \times \mathcal{F}^2_{t_2})_{t_2\ge 0}$*-martingale.*

5 Coupling Measures

Let $\mathcal{C} := \bigcup_{T\in(0,\infty]} C([0,T))$. The map $T : \mathcal{C} \to (0,\infty]$ is such that $[0, T(\xi))$ is the definition domain of ξ. For $t \in [0,\infty)$, let $\mathcal{F}_t$ be the σ-algebra on $\mathcal{C}$ generated by $\{T > s, \xi(s) \in A\}$, where $s \in [0,t]$ and A is a Borel set on $\mathbb{R}$. Then $(\mathcal{F}_t)$ is a filtration on $\mathcal{C}$, and T is an $(\mathcal{F}_t)$-stopping time. Let $\mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$.

For $\xi \in \mathcal{C}$, let $K_\xi(t)$, $0 \le t < T(\xi)$, denote the chordal Loewner hulls driven by ξ. Let H be a hull in $\mathbb{H}$ w.r.t. ∞. Let $T_H(\xi) \in [0, T(\xi)]$ be the maximal number such that $K_\xi(t) \cap \overline{\mathbb{H} \setminus H} = \emptyset$ for $0 \le t < T_H$. Then T_H is an $(\mathcal{F}_t)$-stopping time. Let $\mathcal{C}_H = \{T_H > 0\}$. Then $\xi \in \mathcal{C}_H$ iff H contains some neighborhood of $\xi(0)$ in $\mathbb{H}$. Define $P_H : \mathcal{C}_H \to \mathcal{C}$ such that $P_H(\xi)$ is the restriction of ξ to $[0, T_H(\xi))$. Then $P_H(\mathcal{C}_H) = \{T_H = T\}$, and $P_H \circ P_H = P_H$. If A is a Borel set on $\mathbb{R}$ and $s \in [0,\infty)$, then

$$P_H^{-1}(\{\xi \in \mathcal{C} : T(\xi) > s, \xi(s) \in A\}) = \{\xi \in \mathcal{C}_H : T_H(\xi) > s, \xi(s) \in A\} \in \mathcal{F}_{T_H^-}.$$

Thus, P_H is $(\mathcal{F}_{T_H^-}, \mathcal{F}_\infty)$-measurable on $\mathcal{C}_H$. On the other hand, the restriction of $\mathcal{F}_{T_H^-}$ to $\mathcal{C}_H$ is the σ-algebra generated by $\{\xi \in \mathcal{C}_H : T_H(\xi) > s, \xi(s) \in A\}$, where $s \in [0,\infty)$ and A is a Borel set on $\mathbb{R}$. Thus, $P_H^{-1}(\mathcal{F}_\infty)$ agrees with the restriction of $\mathcal{F}_{T_H^-}$ to $\mathcal{C}_H$.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with spherical metric. Let $\Gamma_{\widehat{\mathbb{C}}}$ denote the space of nonempty compact subsets of $\widehat{\mathbb{C}}$ endowed with Hausdorff metric. Then $\Gamma_{\widehat{\mathbb{C}}}$ is a compact metric space. Define $G : \mathcal{C} \to \Gamma_{\widehat{\mathbb{C}}}$ such that $G(\xi)$ is the spherical closure of $\{t + i\xi(t) : 0 \le t < T(\xi)\}$. Then G is a one-to-one map. Let $I_G = G(\mathcal{C})$. Let $\mathcal{F}^H_{I_G}$ denote the σ-algebra on I_G generated by Hausdorff metric. Let

$$\mathcal{R} = \{\{z \in \mathbb{C} : a < \operatorname{Re} z < b, c < \operatorname{Im} z < d\} : a, b, c, d \in \mathbb{R}\}.$$

Then $\mathcal{F}^H_{I_G}$ agrees with the σ-algebra on I_G generated by $\{\{F \in I_G : F \cap R \ne \emptyset\} : R \in \mathcal{R}\}$. Using this result, one may check that G and G^{-1} (defined on I_G) are both measurable with respect to $\mathcal{F}_\infty$ and $\mathcal{F}^H_{I_G}$.

Now we adopt the notation in the previous section. Let μ_j denote the distribution of $(\xi_j(t), 0 \le t < T_j)$, which is a probability measure on $\mathcal{C}$. Let $\mu = \mu_1 \times \mu_2$ be a probability measure on $\mathcal{C}^2$. Since ξ_1 and ξ_2 are independent, so μ is the joint distribution of ξ_1 and ξ_2.

Let HP_* be the set of $(H_1, H_2) \in \mathrm{HP}$ such that for $j = 1, 2$, H_j is a polygon whose vertices have rational coordinates. Then HP_* is countable. Let (H^m_1, H^m_2), $m \in \mathbb{N}$, be an enumeration of HP_*. For each $n\in\mathbb{N}$, let $M_*^n(t_1,t_2)$ be the $M_*(t_1,t_2)$ given by Theorem 4.2 for (H_1^m,H_2^m), $1\le m\le n$, in the above enumeration. For each $n\in\mathbb{N}$ define $\nu^n=(\nu_1^n,\nu_2^n)$ such that $d\nu^n/d\mu=M_*^n(\infty,\infty)$. From Theorem 4.2, $M_*^n(\infty,\infty)>0$ and $\int M_*^n(\infty,\infty)\,d\mu=\mathbf{E}_\mu[M_*^n(\infty,\infty)]=1$, so ν^n is a probability measure on $\mathcal{C}^2$. Since $d\nu_1^n/d\mu_1=\mathbf{E}_\mu[M_*^n(\infty,\infty)|\mathcal{F}_\infty^1]=M_*^n(\infty,0)=1$, so $\nu_1^n=\mu_1$. Similarly, $\nu_2^n=\mu_2$. So each ν^n is a coupling of μ_1 and μ_2.

Let $\bar\nu^n=(G\times G)_*(\nu^n)$ be a probability measure on $\Gamma_{\hat{\mathbb{C}}}^2$. Since $\Gamma_{\hat{\mathbb{C}}}^2$ is compact, so $(\bar\nu^n)$ has a subsequence $(\bar\nu^{n_k})$ that converges weakly to some probability measure $\bar\nu=(\bar\nu_1,\bar\nu_2)$ on $\Gamma_{\hat{\mathbb{C}}}\times\Gamma_{\hat{\mathbb{C}}}$. Then for $j=1,2$, $\bar\nu_j^{n_k}\to\bar\nu_j$ weakly. For $n\in\mathbb{N}$ and $j=1,2$, since $\nu_j^n=\mu_j$, so $\bar\nu_j^n=G_*(\mu_j)$. Thus $\bar\nu_j=G_*(\mu_j)$, $j=1,2$. So $\bar\nu$ is supported by I_G^2. Let $\nu=(\nu_1,\nu_2)=(G^{-1}\times G^{-1})_*(\bar\nu)$ be a probability measure on $\mathcal{C}^2$. Here we use the fact that G^{-1} is $(\mathcal{F}_{I_G}^H,\mathcal{F}_\infty^j)$-measurable. For $j=1,2$, we have $\nu_j=(G^{-1})_*(\bar\nu_j)=\mu_j$. So ν is also a coupling measure of μ_1 and μ_2.

The following lemma is Lemma 4.1 in [10]. The proof is similar.

Lemma 5.1 *For any $n\in\mathbb{N}$, the restriction of ν to $\mathcal{F}^1_{T_{H_1^n}}\times\mathcal{F}^2_{T_{H_2^n}}$ is absolutely continuous w.r.t. μ, and the Radon-Nikodym derivative is $M(T_{H_1^n}(\xi_1),T_{H_2^n}(\xi_2))$.*

Now suppose that the joint distribution of $\xi_1(t)$, $0\le t<T_1$, and $\xi_2(t)$, $0\le t<T_2$, is the ν in the above lemma instead of $\mu=\mu_1\times\mu_2$. Since the distribution of ξ_j is $\nu_j=\mu_j$, so $\beta_j(t)$, $0\le t<T_j$, is still a chordal SLE$(\kappa;\rho,\kappa-6-\rho)$ trace started from $(x_j;x_j^{\sigma_j},x_k)$. Thus, a.s. $\lim_{t\to T_j^-}\beta_j(t)=x_k$. For $(t_1,t_2)\in\mathcal{D}$, let $B_{j,t_k}(t_j)$ be defined by (4.40). Fix an $(\mathcal{F}_t^k)$-stopping time $\bar t_k\in[0,T_k)$. Choose any $n\in\mathbb{N}$. Let $\bar t_k^n=\bar t_k\wedge T_k(H_k^n)$. Then $\bar t_k^n$ is also an $(\mathcal{F}_t^k)$-stopping time, and satisfies $\bar t_k^n\le T_k(H_k^n)$. From the above lemma and the discussion after Theorem 4.1, we see that $B_{j,\bar t_k^n}(t)$, $0\le t\le T_j(H_j^n)$, is a partial $(\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k^n})_{t\ge0}$-Brownian motion.

Lemma 5.2 *$B_{j,\bar t_k}(t)$, $0\le t<T_j(\bar t_k)$, is a partial $(\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k})_{t\ge0}$-Brownian motion.*

Proof Write T_j^n for $T_j(H_j^n)$, $j=1,2$, $n\in\mathbb{N}$. From the above argument, we know that for any $n\in\mathbb{N}$, $B_{j,\bar t_k^n}(t\wedge T_j^n)$, $0\le t<\infty$, is a continuous $(\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k^n})_{t\ge0}$-martingale. Define $S_j^n=T_j^n$ on $\{\bar t_k\le T_k^n\}$, and $S_j^n=0$ on $\{T_k^n<\bar t_k\}$. Then for any $t\ge0$, $\{S_j^n\le t\}=\{T_k^n<\bar t_k\}\cup\{T_j^n\le t\}\in\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k}$. So S_j^n is an $(\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k})_{t\ge0}$-stopping time. Now we claim that $B_{j,\bar t_k}(t\wedge S_j^n)$, $0\le t<\infty$, is a continuous $(\mathcal{F}_t^j\times\mathcal{F}^k_{\bar t_k})_{t\ge0}$-martingale. Fix $s_2\ge s_1\ge0$ and $\mathcal{E}\in\mathcal{F}^j_{s_1}\times\mathcal{F}^k_{\bar t_k}$. Let $\mathcal{E}_1=\mathcal{E}\cap\{T_k^n<\bar t_k\}$ and $\mathcal{E}_2=\mathcal{E}\cap\{\bar t_k\le T_k^n\}$. Since $S_j^n=0$ on $\mathcal{E}_1$, so $B_{j,\bar t_k}(s_2\wedge S_j^n)=0=B_{j,\bar t_k}(s_1\wedge S_j^n)$ on $\mathcal{E}_1$, which implies that

$$\int_{\mathcal{E}_1}B_{j,\bar t_k}\big(s_2\wedge S_j^n\big)\,d\nu=0=\int_{\mathcal{E}_1}B_{j,\bar t_k}\big(s_1\wedge S_j^n\big)\,d\nu. \tag{5.1}$$

Since $\bar t_k=\bar t_k^n$ on $\{\bar t_k\le T_k^n\}$, so $\mathcal{F}^k_{\bar t_k}$ agrees with $\mathcal{F}^k_{\bar t_k^n}$ on $\{\bar t_k\le T_k^n\}$. Thus, $\mathcal{E}_2\in\mathcal{F}^j_{s_1}\times\mathcal{F}^k_{\bar t_k^n}$. Since $\bar t_k=\bar t_k^n$ and $S_j^n=T_j^n$ on $\mathcal{E}_2$, so from the martingale property of $B_{j,\bar t_k^n}(t\wedge T_j^n)$, we have

$$\int_{\mathcal{E}_2}B_{j,\bar t_k}\big(s_2\wedge S_j^n\big)\,d\nu=\int_{\mathcal{E}_2}B_{j,\bar t_k}\big(s_1\wedge S_j^n\big)\,d\nu. \tag{5.2}$$

Since $\mathcal{E}$ is the disjoint union of $\mathcal{E}_1$ and $\mathcal{E}_2$, so from (5.1) and (5.2), $\mathbf{E}_\nu[B_{j,\bar{t}_k}(s_2 \wedge S_j^n)|\mathcal{F}_{s_1}^j \times \mathcal{F}_{\bar{t}_k}^k] = B_{j,\bar{t}_k}(s_1 \wedge S_j^n)$. So the claim is justified.

Since the above claim holds for any $n \in \mathbb{N}$, so $B_{j,\bar{t}_k}(t)$, $0 \le t < \bigvee_{n=1}^\infty S_j^n$, is a continuous $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)_{t\ge 0}$-local martingale. We now claim that $\bigvee_{n=1}^\infty S_j^n = T_j(\bar{t}_k)$. Fix any $n \in \mathbb{N}$. If $T_k^n < \bar{t}_k$ then $S_j^n = 0 < T_j(\bar{t}_k)$. If $\bar{t}_k \le T_k^n$ then $S_j^n = T_j^n$. From $\bar{t}_k \le T_k^n$ we have $K_k(\bar{t}_k) \subset H_k^n$. From $S_j^n = T_j^n$ we have $K_j(S_j^n) \subset H_j^n$. Since $\overline{H_j^n} \cap \overline{H_k^n} = \emptyset$, so $\overline{K_j(S_j^n)} \cap \overline{K_k(\bar{t}_k)} = \emptyset$, and so again we have $S_j^n < T_j(\bar{t}_k)$. Since the above holds for any $n \in \mathbb{N}$, so $\bigvee_{n=1}^\infty S_j^n \le T_j(\bar{t}_k)$. Now suppose $t_0 < T_j(\bar{t}_k)$. Then $\overline{K_j(t_0)} \cap \overline{K_k(\bar{t}_k)} = \emptyset$. We may always find $(H_1^{n_0}, H_2^{n_0}) \in \mathrm{HP}_*$ such that $K_j(t_0) \subset H_j^{n_0}$ and $K_k(\bar{t}_k) \subset H_k^{n_0}$. Then we have $\bar{t}_k \le T_k^{n_0}$. So $\bigvee_{n=1}^\infty S_j^n \ge S_j^{n_0} = T_j^{n_0} \ge t_0$. Since this holds for any $t_0 < T_j(\bar{t}_k)$, so $\bigvee_{n=1}^\infty S_j^n = T_j(\bar{t}_k)$. Thus, $B_{j,\bar{t}_k}(t)$, $0 \le t < T_j(\bar{t}_k)$, is a continuous $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)_{t\ge 0}$-local martingale. Using a similar argument, we conclude that $B_{j,\bar{t}_k}(t)^2 - t$, $0 \le t < T_j(\bar{t}_k)$, is also a continuous $(\mathcal{F}_t^j \times \mathcal{F}_{\bar{t}_k}^k)_{t\ge 0}$-local martingale. Using the characterization of Brownian motion in [7], we complete the proof. □

Theorem 5.1 *Let $a > 0$. Let $\bar{t}_2 \in (0, T_2)$ be an $(\mathcal{F}_t^2)$-stopping time. Let $C_1 = a \cdot \frac{\xi_2(\bar{t}_2) - p_2(\bar{t}_2)}{p_2(\bar{t}_2) - q_2(\bar{t}_2)} > 0$, $w(z) = C_1 \cdot \frac{z - q_2(\bar{t}_2)}{\xi_2(\bar{t}_2) - z}$, and $W = w \circ \varphi_2(\bar{t}_2, \cdot)$. Then after a time-change, $W(\beta_1(t))$, $0 \le t < T_1(\bar{t}_2)$, has the distribution of a degenerate intermediate* SLE$(\kappa;\rho)$ *trace with force points 0_+ and a. Moreover, a.s. $T_1(\bar{t}_2) < T_1$ and $\beta_1(T_1(\bar{t}_2)) = \beta_2(\bar{t}_2)$.*

Proof Let $C_2 = C_1 \cdot (\xi_2(\bar{t}_2) - q_2(\bar{t}_2)) > 0$. For $0 \le t < T_1(\bar{t}_2)$, define

$$\widetilde{\varphi}(t,z) = \frac{C_2 A_{2,1}(t,\bar{t}_2)}{A_{2,0}(t,\bar{t}_2) - \varphi_{1,\bar{t}_2}(t, w^{-1}(z))} - C_1 + \int_0^t \frac{2C_2 A_{2,1}(s,\bar{t}_2) A_{1,1}(s,\bar{t}_2)^2}{E_{1,0}(s,\bar{t}_2)^3}\, ds; \tag{5.3}$$

$$\widetilde{\xi}(t) = \frac{C_2 A_{2,1}(t,\bar{t}_2)}{E_{2,0}(s,\bar{t}_2)} - C_1 + \int_0^t \frac{2C_2 A_{2,1}(s,\bar{t}_2) A_{1,1}(s,\bar{t}_2)^2}{E_{1,0}(s,\bar{t}_2)^3}\, ds; \tag{5.4}$$

$$\widetilde{p}(t) = \frac{C_2 A_{2,1}(t,\bar{t}_2)}{E_{2,1}(t,\bar{t}_2)} - C_1 + \int_0^t \frac{2C_2 A_{2,1}(s,\bar{t}_2) A_{1,1}(s,\bar{t}_2)^2}{E_{1,0}(s,\bar{t}_2)^3}\, ds; \tag{5.5}$$

$$\widetilde{q}(t) = \frac{C_2 A_{2,1}(t,\bar{t}_2)}{E_{2,2}(t,\bar{t}_2)} - C_1 + \int_0^t \frac{2C_2 A_{2,1}(s,\bar{t}_2) A_{1,1}(s,\bar{t}_2)^2}{E_{1,0}(s,\bar{t}_2)^3}\, ds. \tag{5.6}$$

Since $A_{2,0}(0,\bar{t}_2) = \xi_2(\bar{t}_2)$, $A_{2,1}(0,\bar{t}_2) = 1$, and $\varphi_{1,\bar{t}_2}(0,\cdot) = \mathrm{id}$, so $\widetilde{\varphi}(0,z) = z$. Using (4.15) and (4.16) with $j = 1$ and $k = 2$, it is straightforward to check that

$$\partial_t \widetilde{\varphi}(t,z) = \frac{2C_2^2 N(t,\bar{t}_2)^2}{\widetilde{\varphi}(t,z) - \widetilde{\xi}(t)}. \tag{5.7}$$

Let $v(t) = \int_0^t C_2^2 N(s,\bar{t}_2)^2\, ds$. Then $v(0) = 0$ and v is continuous and strictly increasing. So v maps $[0, T_1(\bar{t}_2))$ onto $[0, T)$ for some $T \in (0,\infty]$. Let $\varphi(t,\cdot) = \widetilde{\varphi}(v^{-1}(t),\cdot)$ and $\xi(t) = \widetilde{\xi}(v^{-1}(t))$ for $0 \le t < T$. From (5.7), we have $\partial_t \varphi(t,z) = \frac{2}{\varphi(t,z) - \xi(t)}$. Thus $\varphi(t,\cdot)$, $0 \le t < T$, are the chordal Loewner maps driven by ξ.

Note that w maps $\mathbb{H}$ conformally onto $\mathbb{H}$, and $w(\xi_2(\bar{t}_2)) = \infty$. Since $\varphi_2(\bar{t}_2,\cdot)$ maps $\mathbb{H} \setminus \beta_2((0,\bar{t}_2])$ conformally onto $\mathbb{H}$, and $\varphi_2(\bar{t}_2, \beta_2(\bar{t}_2)) = \xi_2(\bar{t}_2)$, so W maps $\mathbb{H} \setminus \beta_2((0,\bar{t}_2])$ conformally on $\mathbb{H}$, and $W(\beta_2(\bar{t}_2)) = \infty$. For any $t \in [0, T_1(\bar{t}_2))$, w^{-1} maps $\mathbb{H} \setminus W(\beta_1((0,t]))$

conformally onto $\mathbb{H}\setminus\varphi_2(\bar t_2,\beta_1((0,t]))=\mathbb{H}\setminus K_{1,\bar t_2}(t)$. Since $\varphi_{1,\bar t_2}(t,\cdot)$ maps $\mathbb{H}\setminus K_{1,\bar t_2}(t)$ conformally onto $\mathbb{H}$, so from (5.3), $\widetilde\varphi(t,\cdot)$ maps $\mathbb{H}\setminus W(\beta_1((0,t]))$ conformally onto $\mathbb{H}$. For $0\le t<T$, let $\beta(t)=W(\beta_1(v^{-1}(t)))$, then $\varphi(t,\cdot)$ maps $\mathbb{H}\setminus\beta((0,t])$ conformally onto $\mathbb{H}$. So $\beta(t)$, $0\le t<T$, is the chordal Loewner trace driven by ξ.

Let $p(t)=\widetilde p(v^{-1}(t))$ and $q(t)=\widetilde q(v^{-1}(t))$, $0\le t<T$. Applying (4.15) and (4.19) with $j=1$ and $k=2$, and using $v'(t)=C_2^2N(t,\bar t_2)^2$, it is straightforward to check that

$$p'(t)=\frac{2}{p(t)-\xi(t)},\quad 0<t<T;\qquad q'(t)=\frac{2}{q(t)-\xi(t)},\quad 0\le t\le T. \tag{5.8}$$

Moreover, since $A_{1,0}(t,\bar t_2)<B_{1,0}(t,\bar t_2)<B_{2,0}(t,\bar t_2)<A_{2,0}(t,\bar t_2)$ for $0<t<T_1(\bar t_2)$, so from (5.4)–(5.6) and the definition of $E_{2,m}$, $m=0,1,2$, we have

$$\xi(t)<p(t)<q(t)<\infty,\quad 0<t<T. \tag{5.9}$$

Since $A_{1,0}(0,\bar t_2)=q_2(\bar t_2)=B_{1,0}(0,\bar t_2)$, and $A_{2,0}(0,\bar t_2)=\xi_2(\bar t_2)$, so $E_{2,0}(0,\bar t_2)=E_{2,1}(0,\bar t_2)=\xi_2(\bar t_2)-q_2(\bar t_2)$. Note that $A_{2,1}(0,\bar t_2)=1$, so

$$\xi(0)=p(0)=\frac{C_2}{\xi_2(\bar t_2)-q_2(\bar t_2)}-C_1=0. \tag{5.10}$$

Since $B_{2,0}(0,\bar t_2)=p_2(\bar t_2)$, so $E_{2,2}(0,\bar t_2)=\xi_2(\bar t_2)-p_2(\bar t_2)$. Thus,

$$q(0)=\frac{C_2}{\xi_2(\bar t_2)-p_2(\bar t_2)}-C_1=a>0. \tag{5.11}$$

Note that $E_{2,0}=-E_{1,0}$. Applying (4.15) and (4.21) with $j=1$, $k=2$ and $m=0$, we get

$$\begin{aligned}d\widetilde\xi(t)&=C_2N(t,\bar t_2)\,d\xi_1(t)\\&\quad+C_2\frac{A_{2,1}(t,\bar t_2)}{E_{1,0}(t,\bar t_2)}\left[\left(\frac{\kappa}{2}-3\right)\frac{A_{1,2}(t,\bar t_2)}{E_{1,0}(t,\bar t_2)}+(6-\kappa)\frac{A_{1,1}(t,\bar t_2)^2}{E_{1,0}(t,\bar t_2)^2}\right]dt.\end{aligned} \tag{5.12}$$

From (4.1), (4.39) and (4.40), we see that $\xi_1(t)$, $0\le t<T_1(\bar t_2)$, satisfies the $(\mathcal{F}_t^1\times\mathcal{F}_{\bar t_2}^2)_{t\ge0}$-adapted SDE:

$$\begin{aligned}d\xi_1(t)&=\sqrt{\kappa}\,dB_{1,\bar t_2}(t)\\&\quad+\left[\left(3-\frac{\kappa}{2}\right)\left(\frac{A_{1,2}}{A_{1,1}}-\frac{2A_{1,1}}{E_{1,0}}\right)+g_0(R)\left(\frac{A_{1,1}}{E_{1,1}}-\frac{A_{1,1}}{E_{1,2}}\right)\right]\Bigg|_{(t,\bar t_2)}dt.\end{aligned} \tag{5.13}$$

From (5.12) and (5.13) we conclude that

$$d\widetilde\xi(t)=C_2N(t,\bar t_2)\left[\sqrt{\kappa}\,dB_{1,\bar t_2}(t)+g_0(R(t,\bar t_2))\left(\frac{A_{1,1}(t,\bar t_2)}{E_{1,1}(t,\bar t_2)}-\frac{A_{1,1}(t,\bar t_2)}{E_{1,2}(t,\bar t_2)}\right)dt\right]. \tag{5.14}$$

Let

$$S(t)=\frac{g_0(R(t,\bar t_2))}{C_2N(t,\bar t_2)}\left(\frac{A_{1,1}(t,\bar t_2)}{E_{1,1}(t,\bar t_2)}-\frac{A_{1,1}(t,\bar t_2)}{E_{1,2}(t,\bar t_2)}\right). \tag{5.15}$$

Since $\widetilde\xi(t)=\xi(v(t))$ and $v'(t)=C_2^2N(t,\bar t_2)^2$, so from (5.14) and Lemma 5.2, there is a Brownian motion $B(t)$ such that for $0<t<T$,

$$d\xi(t)=\sqrt{\kappa}\,dB(t)+S(v^{-1}(t))\,dt. \tag{5.16}$$

From (5.4)–(5.6), we have

$$\widetilde{p}(t)-\widetilde{\xi}(t)=C_2\,\frac{A_{2,1}(t,\bar{t}_2)E_{1,1}(t,\bar{t}_2)}{E_{1,0}(t,\bar{t}_2)E_{2,1}(t,\bar{t}_2)};$$

$$\widetilde{q}(t)-\widetilde{\xi}(t)=C_2\,\frac{A_{2,1}(t,\bar{t}_2)E_{1,2}(t,\bar{t}_2)}{E_{1,0}(t,\bar{t}_2)E_{2,2}(t,\bar{t}_2)}.$$

Thus,

$$\frac{\widetilde{p}(t)-\widetilde{\xi}(t)}{\widetilde{q}(t)-\widetilde{\xi}(t)}=\frac{E_{1,1}(t,\bar{t}_2)E_{2,2}(t,\bar{t}_2)}{E_{1,2}(t,\bar{t}_2)E_{2,1}(t,\bar{t}_2)}=R(t,\bar{t}_2).$$

From (3.7), (5.15) and the above formulas, we get

$$J(\widetilde{p}(t)-\widetilde{\xi}(t),\widetilde{q}(t)-\widetilde{\xi}(t))=-\left(\frac{1}{\widetilde{p}(t)-\widetilde{\xi}(t)}-\frac{1}{\widetilde{q}(t)-\widetilde{\xi}(t)}\right)\cdot g_0\left(\frac{\widetilde{p}(t)-\widetilde{\xi}(t)}{\widetilde{q}(t)-\widetilde{\xi}(t)}\right)=S(t).$$

From (5.16) we find that, for $0<t<T$,

$$d\xi(t)=\sqrt{\kappa}\,dB(t)+J(p(t)-\xi(t),q(t)-\xi(t))\,dt. \tag{5.17}$$

So $\xi(t)$, $p(t)$ and $q(t)$, $0<t<T$, solve (5.8) and (5.17), and satisfy (5.9)–(5.11). Assume that this solution can be extended beyond T. Since $\kappa\in(0,4)$, so $\beta(T)=\lim_{t\to T^-}\beta(t)\in\mathbb{H}$. Thus, $\lim_{t\to(T_1(\bar{t}_2))^-}W(\beta_1(t))\in\mathbb{H}$. From the definition, W maps $\mathbb{H}\setminus\beta((0,\bar{t}_2])$ conformally onto $\mathbb{H}$. So we have $\lim_{t\to(T_1(\bar{t}_2))^-}\beta_1(t)\in\mathbb{H}\setminus\beta((0,\bar{t}_2])$. This implies that the distance between $\beta_1((0,T_1(\bar{t}_2)])$ and $\beta_2((0,\bar{t}_2])$ is positive. This is impossible because of the definition of $T_1(\bar{t}_2)$ and the fact that $\lim_{t\to T_1^-}\beta_1(t)=x_2=\beta_2(0)$. Thus $(0,T)$ is the maximal interval of the solution. From (5.8)–(5.11) and (5.17), we see that $\beta(t)$, $0\le t<T$, is a degenerate intermediate SLE$(\kappa;\rho)$ trace with force points 0^+ and a. Since β is a time-change of $W(\beta_1)$, so after a time-change, $W(\beta_1(t))$, $0\le t<T_1(\bar{t}_2)$, has the distribution of a degenerate intermediate SLE$(\kappa;\rho)$ trace with force points 0_+ and a.

From Corollary 3.1 and the fact that $W^{-1}(\infty)=\beta_2(\bar{t}_2)$, we see that a.s. $\beta_2(\bar{t}_2)$ is a subsequential limit of $\beta_1(t)$ as $t\to(T_1(\bar{t}_2))^-$. If $T_1(\bar{t}_2)=T_1$ then $\lim_{t\to(T_1(\bar{t}_2))^-}\beta_1(t)=\lim_{t\to T_1^-}\beta_1(t)=x_2\neq\beta_2(\bar{t}_2)$ because $\bar{t}_2>0$, which a.s. does not happen. Thus, a.s. $T_1(\bar{t}_2)<T_1$. Since β_1 is continuous on $[0,T_1)$, so a.s. $\beta_1(T_1(\bar{t}_2))=\lim_{t\to(T_1(\bar{t}_2))^-}\beta_1(t)$. Since a.s. $\beta_2(\bar{t}_2)$ is a subsequential limit of $\beta_1(t)$ as $t\to(T_1(\bar{t}_2))^-$, so $\beta_1(T_1(\bar{t}_2))=\beta_2(\bar{t}_2)$. □

Theorem 5.2 *Almost surely* $\beta_1((0,T_1))=\beta_2((0,T_2))$.

Proof For $n\in\mathbb{N}$, let S_n be the first time that $|\beta_2(t)-x_1|=|x_2-x_1|/(n+1)$. Then for each $n\in\mathbb{N}$, S_n is an $(\mathcal{F}^2_t)$-stopping time, $S_n\in(0,T_2)$, and $T_2=\bigvee_{n=1}^\infty S_n$. For each $q\in\mathbb{Q}_{>0}$, let $S_{n,q}=S_n\wedge q$, which is also an $(\mathcal{F}^2_t)$-stopping time. Then $\{S_{n,q}:n\in\mathbb{N},q\in\mathbb{Q}_{>0}\}$ is a dense subset of $(0,T)$. Applying Theorem 5.1 with $\bar{t}_2=S_{n,q}$, we see that a.s. $\beta_2(S_{n,q})\in\beta_1((0,T_1))$ for any $n\in\mathbb{N}$ and $q\in\mathbb{Q}_{>0}$. From the denseness of $\{S_{n,q}\}$ and the continuity of β_1, we have a.s. $\beta_2((0,T_2))\subset\beta_1((0,T_1))$. Since both β_1 and β_2 are simple curves, $\beta_1(0)=x_1=\beta_2(T_2)$, and $\beta_2(0)=x_2=\beta_1(T_1)$, so a.s. $\beta_1((0,T_1))=\beta_2((0,T_2))$. □

Corollary 5.1 *Suppose* $\beta(t)$, $0\le t<\infty$, *is a degenerate intermediate* SLE$(\kappa;\rho)$ *trace. Then a.s.* $\lim_{t\to\infty}\beta(t)=\infty$.

Proof Suppose that the force points for β is 0^+ and $a_0>0$. Applying Theorem 5.1 with $a=a_0$ and any $(\mathcal{F}^2_t)$-stopping time $\bar t_2\in(0,T_2)$. Then $W(\beta_1(t))$, $0\le t<T_1(\bar t_2)$, has the same distribution as $\beta(t)$, $0\le t<\infty$, up to a time-change, and a.s. $\lim_{t\to(T_1(\bar t_2))^-}\beta_1(t)=\beta_1(T_1(\bar t_2))=\beta_2(\bar t_2)$. Since $W(\beta_2(\bar t_2))=\infty$, so a.s. $\lim_{t\to(T_1(\bar t_2))^-}W(\beta_1(t))=\infty$. Thus, a.s. $\lim_{t\to\infty}\beta(t)=\infty$. □

Proof of Theorem 1.1 We may find W_1 that maps $\mathbb{H}$ conformally or conjugate conformally onto $\mathbb{H}$ such that $W_1(x_1)=0$, $W_1(x_1^+)=0^\sigma$, and $W_1(x_2)=\infty$. Let $W_2=W_0^{-1}\circ W_1$. Then W_2 maps $\mathbb{H}$ conjugate conformally or conformally onto $\mathbb{H}$ such that $W_2(x_2)=0$, $W_2(x_2^-)=0^\sigma$, and $W_2(x_1)=\infty$. Recall that for $j=1,2$, $\beta_j(t)$, $0<t<T_j$, is a chordal SLE($\kappa;\rho,\kappa-6-\rho$) trace started from $(x_j;x_j^{\sigma_j},x_{3-j})$, where $\sigma_1=+$ and $\sigma_2=-$. From Proposition 2.1, after a time-change, $W_j^{-1}(\beta_0(t))$, $0<t<\infty$, has the same distribution as $\beta_j(t)$, $0<t<T_j$, $j=1,2$. From Theorem 5.1, after a time-change, the reversal of $\beta_2(t)$, $0<t<T_2$, agrees with $\beta_1(t)$, $0<t<T_1$. Thus, $W_2^{-1}(\beta_0(1/t))$, $0<t<\infty$, has the same distribution as $W_1^{-1}(\beta_0(t))$, $0<t<\infty$, after a time-change. Since $W_0=W_1\circ W_2^{-1}$, so the proof is finished. □

Proof of Theorem 1.2 Applying Theorem 5.1 with any $(\mathcal{F}^2_t)$-stopping time $\bar t_2\in(0,T_2)$ and $a=1/b_0$, we get $w(z)=a\cdot\frac{\xi_2(\bar t_2)-p_2(\bar t_2)}{p_2(\bar t_2)-q_2(\bar t_2)}\cdot\frac{z-q_2(\bar t_2)}{\xi_2(\bar t_2)-z}$ and $W=w\circ\varphi_2(\bar t_2,\cdot)$, such that after a time-change, $W(\beta_1(t))$, $0\le t<T_1(\bar t_2)$, has the same distribution as a degenerate intermediate SLE($\kappa;\rho$) trace with force points 0^+ and $a=1/b_0$.

Let $\widetilde T=T_2-\bar t_2$. For $0\le t<\widetilde T$, let $\widetilde\xi(t)=\xi_2(\bar t_2+t)$, $\widetilde p(t)=p_2(\bar t_2+t)$ and $\widetilde q(t)=q_2(\bar t_2+t)$. Let $\widetilde B(t)=B_2(\bar t_2+t)-B_2(\bar t_2)$, $t\ge0$. Then $\widetilde B(t)$ is a Brownian motion that is independent of $\xi_2(\bar t_2)$, $p_2(\bar t_2)$ and $q_2(\bar t_2)$. From (4.1)–(4.3), $\widetilde\xi(t)$, $\widetilde p(t)$ and $\widetilde q(t)$, $0\le t<\widetilde T$, satisfy the following SDE:

$$d\widetilde\xi(t)=\sqrt{\kappa}\,d\widetilde B(t)+\frac{\rho}{\widetilde\xi(t)-\widetilde p(t)}\,dt+\frac{\kappa-6-\rho}{\widetilde\xi(t)-\widetilde q(t)}\,dt,$$

$$d\widetilde p(t)=\frac{2}{\widetilde p(t)-\widetilde\xi(t)}\,dt,\qquad d\widetilde q(t)=\frac{2}{\widetilde q(t)-\widetilde\xi(t)}\,dt,$$

with initial values

$$\widetilde\xi(0)=\xi_2(\bar t_2),\qquad \widetilde p(0)=p_2(\bar t_2),\qquad \widetilde q(0)=q_2(\bar t_2).$$

For $0\le t<\widetilde T$, let $\widetilde\varphi(t,\cdot)=\varphi_2(\bar t_2+t,\cdot)\circ\varphi_2(\bar t_2,\cdot)^{-1}$ and $\widetilde\beta(t)=\varphi_2(\bar t_2,\beta_2(\bar t_2+t))$. Then $\widetilde\varphi(0,z)=z$, and $\widetilde\varphi(t,z)$, $0\le t<\widetilde T$, satisfy $\partial_t\widetilde\varphi(t,z)=\frac{2}{\widetilde\varphi(t,z)-\widetilde\xi(t)}$, and for each $0\le t<\widetilde T$, $\widetilde\varphi(t,\cdot)$ maps $\mathbb{H}\setminus\widetilde\beta((0,t])$ conformally onto $\mathbb{H}$. Thus, $\widetilde\beta(t)$, $0\le t<\widetilde T$, is the chordal Loewner trace driven by $\widetilde\xi$. The solution $\widetilde\xi(t)$, $\widetilde p(t)$ and $\widetilde q(t)$, $0\le t<\widetilde T$, could not be extended beyond $\widetilde T$ because $\lim_{t\to\widetilde T^-}\widetilde\beta(t)=\varphi_2(\bar t_2,\lim_{t\to T_2^-}\beta_2(t))=\varphi_2(\bar t_2,x_1)\in\mathbb{R}$. Thus, $\widetilde\beta(t)=\varphi_2(\bar t_2,\beta_2(\bar t_2+t))$, $0<t<T_2-\bar t_2$, is a chordal SLE($\kappa;\rho,\kappa-6-\rho$) trace started from $(\xi_2(\bar t_2);p_2(\bar t_2),q_2(\bar t_2))$. Let $W_1=W_0^{-1}\circ w$. Then $W_0^{-1}\circ W=W_1\circ\varphi_2(\bar t_2,\cdot)$, W_1 maps $\mathbb{H}$ conformally onto $\mathbb{H}$, $W_1(\xi_2(\bar t_2))=0$, $W_1(q_2(\bar t_2))=\infty$ and $W_1(p_2(\bar t_2))=1/a=b_0$. From Proposition 2.1, $W_0^{-1}\circ W(\beta_2(\bar t_2+t))=W_1(\widetilde\beta(t))$, $0<t<T_2-\bar t_2$, has the same distribution as $\beta_0(t)$, $0<t<\infty$, after a time-change. From Theorems 5.1 and 5.2, after a time-change, the reversal of $\beta_2(t)$, $\bar t_2<t<T_2$, has the same distribution as $\beta_1(t)$, $0<t<T_1(\bar t_2)$. Thus, after a time-change, $W_0(\beta_0(1/t))$, $0<t<\infty$, has the same distribution as the reversal of $W(\beta_1(t))$, $0<t<T_1(\bar t_2)$, which has the same distribution as a degenerate intermediate SLE($\kappa;\rho$) trace with force points 0^+ and $1/b_0$. □

Now we will see some applications of Theorem 1.1. The following proposition is Theorem 5.4 in [10], where $\partial_{\mathbb{H}}^{+} S$ is the right boundary of S in $\mathbb{H}$ (cf. [10]).

Proposition 5.1 *Let* $\kappa > 4$, $C \ge 1/2$, *and* $K(t)$, $0 \le t < \infty$, *be a chordal* $\mathrm{SLE}(\kappa; C(\kappa - 4))$ *process started from* $(0; 0^+)$. *Let* $K(\infty) = \bigcup_{t<\infty} K(t)$. *Let* $W_0(z) = 1/\overline{z}$. *Then* $W_0(\partial_{\mathbb{H}}^{+} K(\infty))$ *has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; C'(\kappa' - 4), \frac{1}{2}(\kappa' - 4))$ *trace started from* $(0; 0^+, 0^-)$, *where* $\kappa' = 16/\kappa$ *and* $C' = 1 - C$.

Applying the above proposition with $C = 1$, and applying Theorem 1.1 with $\kappa = \kappa'$ and $\rho = \frac{1}{2}(\kappa' - 4)$, we conclude the following theorem, which is Conjecture 2 in [1].

Theorem 5.3 *Let* $\kappa > 4$, *and* $K(t)$, $0 \le t < \infty$, *be a chordal* $\mathrm{SLE}(\kappa; \kappa - 4)$ *process started from* $(0; 0^+)$. *Let* $K(\infty) = \bigcup_{t<\infty} K(t)$. *Then* $\partial_{\mathbb{H}}^{+} K(\infty)$ *has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; \frac{1}{2}(\kappa' - 4))$ *trace started from* $(0; 0^-)$, *where* $\kappa' = 16/\kappa$.

The following proposition is a part of Theorem 5.2 in [11].

Proposition 5.2 *Let* $\kappa > 4$ *and* $C_+, C_- \ge 1/2$. *Let* $K(t)$, $0 \le t < \infty$, *be a chordal* $\mathrm{SLE}(\kappa; C_+(\kappa - 4), C_-(\kappa - 4))$ *process started from* $(0; 0^+, 0^-)$. *Let* $K(\infty) = \bigcup_{t \ge 0} K(t)$. *Let* $\kappa' = 16/\kappa$ *and* $W_0(z) = 1/\overline{z}$. *Then* $W_0(\partial_{\mathbb{H}}^{+} K(\infty))$ *has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; (1 - C_+)(\kappa' - 4), (1/2 - C_-)(\kappa' - 4))$ *trace started from* $(0; 0^+, 0^-)$.

Applying Proposition 5.2 with $C+ = 1$ or $C- = 1/2$, and using Theorem 1.1, we conclude the following two theorems.

Theorem 5.4 *Let* $\kappa > 4$, $C \ge 1/2$, *and* $K(t)$, $0 \le t < \infty$, *be a chordal* $\mathrm{SLE}(\kappa; \kappa - 4, C(\kappa - 4))$ *process started from* $(0; 0^+, 0^-)$. *Let* $K(\infty) = \bigcup_{t<\infty} K(t)$. *Then* $\partial_{\mathbb{H}}^{+} K(\infty)$ *has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; (1/2 - C)(\kappa' - 4))$ *trace started from* $(0; 0^-)$, *where* $\kappa' = 16/\kappa$.

Theorem 5.5 *Let* $\kappa > 4$, $C \ge 1/2$, *and* $K(t)$, $0 \le t < \infty$, *be a chordal* $\mathrm{SLE}(\kappa; C(\kappa - 4), \frac{1}{2}(\kappa - 4))$ *process started from* $(0; 0^+, 0^-)$. *Let* $K(\infty) = \bigcup_{t<\infty} K(t)$. *Then* $\partial_{\mathbb{H}}^{+} K(\infty)$ *has the same distribution as the image of a chordal* $\mathrm{SLE}(\kappa'; (1 - C)(\kappa' - 4))$ *trace started from* $(0; 0^+)$, *where* $\kappa' = 16/\kappa$.

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